CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 7**

Model Solution No: 1

Make y the subject of the first equation to get $y = 1 - 5x$. Then substituting it into the second we obtain:

$$
\begin{aligned}
x^2 + (y-2)^2 = 1 &\Rightarrow x^2 + (1 - 5x - 2)^2 = 1 \\
&\Rightarrow x^2 + (-1 - 5x)^2 = 1 \\
&\Rightarrow x^2 + (5x + 1)^2 = 1 \\
&\Rightarrow x^2 + 25x^2 + 10x + 1 = 1 \\
&\Rightarrow 26x^2 + 10x = 0 \\
&\Rightarrow x(26x + 10) = 0 \\
&\Rightarrow x = 0 \quad \text{or} \quad x = -\frac{5}{13}
\end{aligned}
$$

and substituting back in, you will be able to find the values of y (details left as an exercise)

Answer: $x = 0, y = 1$ or $x = -\frac{5}{13}, y = \frac{38}{13}$

Extension: e.g. it means there are two intersections between the line $5x + y = 1$ and the circle with centre $(0, 2)$ and radius 1

Question Sheet: **Sheet 7**

Model Solution No: 2

(a) (i) **Solution:** Let a and b be consecutive integers. Then either a is even or b is even. Suppose (without loss of generality) that a is even, then $a = 2k$ for some integer k and $b = 2k + 1$ (by definition).

Thus we have

$$ab = 2k(2k + 1) = 2 \times k(k + 1)$$

which is even as required.

(a) (ii) **Solution:** $n^2 + n + 1 = n(n + 1) + 1$. If n is an integer, then $n(n + 1)$ is even (from part (a)), and hence $n(n + 1) + 1$ is odd as required.

(b) **Solution:** Let $n = 3$, then $n^2 - 1 = 8$, which is not prime.

Extension: the method is to consider the cases of even and odd integers separately.

Let $n = 2k$ for some integer k, then $n^2+n+1 = (2k)^2+2k+1 = 4k^2+2k+1 = 2(2k^2+k)+1$, which is odd. Thus if n is even, then $n^2 + n + 1$ is odd.

Alternatively, let n be odd, i.e. $n = 2p + 1$ for some integer p. Then

$$\begin{aligned} n^2 + n + 1 &= (2p + 1)^2 + (2p + 1) + 1 \\ &= 4p^2 + 4p + 1 + 2p + 1 + 1 \\ &= 4p^2 + 6p + 3 \\ &= 2(2p^2 + 3p + 1) + 1 \end{aligned}$$

which is odd.

Hence if n is an integer, then $n^2 + n + 1$ is odd.

Question Sheet: **Sheet 7**

Model Solution No: 3

(a) Integrating the equation once, we get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \int (12x - 4)\,\mathrm{d}x \\ &= 6x^2 - 4x + c\end{aligned}$$

We know that there is a stationary point at $x = 2$, so we must have that

$$0 = 6(2)^2 - 4(2) + c \Rightarrow c = -16$$

Now can integrate again to find y:

$$\begin{aligned}y &= \int (6x^2 - 4x - 16)\,\mathrm{d}x \\ &= 2x^3 - 2x^2 - 16x + k\end{aligned}$$

We know that $(2, -1)$ lies on C, so

$$-1 = 2(2)^3 - 2(2)^2 - 16(2) + k \Rightarrow k = 23$$

Hence $y = 2x^3 - 2x^2 - 16x + 23$

Answer: $y = 2x^3 - 2x^2 - 16x + 23$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 4x - 16$

Stationary points are when $6x^2 - 4x - 16 = 0$. Dividing both sides by 2 and factorising, we have $(3x + 4)(x - 2) = 0$.

Thus the other stationary point has $x = -\frac{4}{3}$. Substituting back in to find y, we obtain the coordinates $\left(-\frac{4}{3}, \frac{973}{27}\right)$

When $x = -\frac{4}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12\left(-\frac{4}{3}\right) - 4 = -20$. This is less than 0, so the point is a maximum
Answer: maximum at $\left(-\frac{4}{3}, \frac{973}{27}\right)$

Question Sheet: **Sheet 7**

Model Solution No: 4

(a) **Solution:** $\log_4(64x) = \log_4(64) + \log_4(x) = 3 + p$

(b) **Solution:** $\log_4\left(\frac{x^3}{4}\right) = 3\log_4(x) - \log_4 4 = 3p - 1$

(c) Using the previous parts, the equation becomes

$$(3 + p) - (3p - 1) = 3$$

which gives $-2p = -1$, so $p = \frac{1}{2}$

Reversing the substitution, we have $\log_4 x = \frac{1}{2} \Rightarrow x = 4^{\frac{1}{2}} = 2$

Question Sheet: **Sheet 7**

Model Solution No: 5

(a) **Solution:** $\ln D = \ln(ae^{bT}) \Rightarrow \ln D = \ln a + \ln(e^{bT}) = \ln a + bT$

Thus $m = b$ and $c = \ln a$

(b) The gradient of the line is $\frac{2.4-1.8}{0-4} = -0.15$. This corresponds to the value of b

The intercept is 2.4 which is $\ln a$, thus $a = e^{2.4}$

Answer: $a = e^{2.4}$, $b = -0.15$ (or awrt to 11 for a)

(c) Our model is $D = e^{2.4} \times e^{-0.15T}$. At $T = 5$, the amount of drug in the body due to the initial dose is $D = 5.206....$ Hence immediately after the second dose, the patient has $5.206... + 7 = 12.206...$ mg of drug in their system

Answer: awrt 12.2 mg

(d) **Solution:** Higher initial dose means a is greater, so the graph for patient B will have a higher y intercept.

Patient B breaks the drug down at a faster rate so b is smaller, which means that the gradient of the line for patient B will be more negative.

Question Sheet: **Sheet 7**

Model Solution No: 6

(a) **Solution:** Consider the whole system and resolve to the right to obtain:

$$30 - k(1) - k(2) - k(4) = (1 + 2 + 4)(3)$$

This re-arranges to $7k = 9$, so $k = \frac{9}{7}$ as desired.

(b) For example, consider particle P and resolve to the right to get

$$30 - T_{PQ} - \frac{9}{7}(1) = (1)(3)$$

which gives $T_{PQ} = \frac{180}{7}$

Answer: $T_{PQ} = \frac{180}{7}$ N

(c) Now consider particle R and resolve to the right again to get

$$T_{RQ} - \frac{9}{7}(4) = 4(3)$$

which gives $T_{RQ} = \frac{120}{7}$

Answer: $T_{RQ} = \frac{120}{7}$ N

(d) **Answer:** e.g. the acceleration of all three particles is the same

(e) **Solution:** Consider the equation of motion for P once the string breaks:

$$30 - \frac{9}{7}(1) = 1a$$

The main thing to notice is that $a > 0$, so the particle P will continue to move indefinitely with constant acceleration $a = \frac{201}{7}$ (by Newton's first law unless some additional change to the system is made).

Question Sheet: **Sheet 7**

Model Solution No: 7

(a) **Answer:** overseas, because the temperatures are too high for a UK location. [NB: the location is Beijing and you should be able to identify this too]

(b) From the data, the probability of a day having a temperature greater than 20°C is $\frac{136}{184} = \frac{17}{23}$.

A binomial distribution is suitable here, so use $X \sim B\left(50, \frac{17}{23}\right)$

Answer: $X \sim B\left(50, \frac{17}{23}\right)$ [NB: any uppercase random variable letter is suitable]

(c) (i) **Answer:** $\mathbb{P}(X = 38) = 0.124$ (3 sf).

(c) (ii) **Answer:** $\mathbb{P}(X \geq 40) = 0.209$ (3 sf).

(d) **Answer:** e.g. LDS only contains data from March-October, so her probability model cannot be applied (reliably) outside this range

Extension: cloud cover/wind speed on the Beaufort scale

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