CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 4**

Model Solution No: 1

Solution: The limit definition of the derivative is that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$$

Using this with $y = 2x^3 - x + 2$ gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0} \frac{2(x+h)^3 - (x+h) + 2 - (2x^3 - x + 2)}{h} \\
&= \lim_{h\to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - x - h + 2 - 2x^3 + x - 2}{h} \\
&= \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3 - h}{h} \\
&= \lim_{h\to 0} (6x^2 + 6xh + 2h^2 - 1) \\
&= 6x^2 - 1
\end{aligned}$$

as required.

Question Sheet: **Sheet 3**

Model Solution No: 2

First note $\sqrt{45} = 3\sqrt{5}$.

Second note $(1-\sqrt{5})^2 = 1 - 2\sqrt{5} + 5 = 6 - 2\sqrt{5}$

Then our expression becomes

$$\frac{1+3\sqrt{5}}{6-2\sqrt{5}}$$

We can express this in the required form by rationalising the denominator. This gives

$$\begin{aligned}
\frac{1+3\sqrt{5}}{6-2\sqrt{5}} &= \frac{(1+3\sqrt{5})(6+2\sqrt{5})}{(6-2\sqrt{5})(6+2\sqrt{5})} \\
&= \frac{6+2\sqrt{5}+18\sqrt{5}+6(5)}{36-20} \\
&= \frac{36+20\sqrt{5}}{16} \\
&= \frac{9+5\sqrt{5}}{4} \\
&= \frac{9}{4}+\frac{5}{4}\sqrt{5}
\end{aligned}$$

so $a = \frac{9}{4}$ and $b = \frac{5}{4}$

Answer: $\frac{9}{4}+\frac{5}{4}\sqrt{5}$ (or equivalent)

Question Sheet: **Sheet 4**

Model Solution No: 3

Using $\tan x = \dfrac{\sin x}{\cos x}$, we have

$$8 \sin x = \frac{5 \sin x}{\cos x}$$

which re-arranges to $\sin x(8 \cos x - 5) = 0$.

This is true either if $\sin x = 0$ or $8 \cos x - 5 = 0$.

In the first case, $(\sin x = 0)$ we have $x = 0,\ 180,\ 360$ for our interval.

The second case gives $\cos x = \frac{5}{8}$. The principal value is then $x_{\text{principal}} = 51.317$ and the other relevant value in range is $360 - 51.317 = 308.683$.

Answer: 0, 51.3, 180, 309, 360

Question Sheet: **Sheet 4**

Model Solution No: 4

The gradient of l_1 is

$$m_{l_1} = \frac{8-6}{-2-4} = -\frac{1}{3}$$

Since l_2 is perpendicular to this line, it must have gradient 3. Furthermore, since it passes through $(1, 4)$, its equation is

$$y - 4 = 3(x - 1)$$

The points P and Q occur when $x = 0$ and $y = 0$ respectively (or you can define them the other way around).

So for P, we substitute in $x = 0$ which gives $y = 1$. Then, for Q, we substitute in $y = 0$ gives $x = -\frac{1}{3}$.

So $P = (0, 1)$ and $Q = (-\frac{1}{3}, 0)$.

Now we know that PQ is a diameter of the circle C, so its midpoint is the centre of C and its length is twice the radius. It is left as an exercise for you to verify that the centre is thus $(-\frac{1}{6}, \frac{1}{2})$ and the radius is $\frac{\sqrt{10}}{6}$. The equation of C then follows as

$$\left(x + \frac{1}{6}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{18}$$

Answer: $\left(x + \frac{1}{6}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{18}$

Question Sheet: **Sheet 4**

Model Solution No: 5

(a) **Solution:** Since the area is 120 cm^2, we have that

$$120 = 2\pi r^2 + 2\pi r h$$

which re-arranges to $h = \frac{120-2\pi r^2}{2\pi r}$.

The volume V of the cylinder thus satisfies

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 \left(\frac{120 - 2\pi r^2}{2\pi r} \right) \\ &= 60r - \pi r^3 \end{aligned} \tag{1}$$

as required.

(b) We have $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 60 - 3\pi r^2$.

When V is stationary, $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 0$. So at the stationary point, we must have

$$60 - 3\pi r^2 = 0$$

or $r = \sqrt{\frac{60}{3\pi}} \quad (r > 0)$

Answer: $r = \sqrt{\frac{20}{\pi}}$.

(c) **Solution:** $\dfrac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi r$

When $r = \sqrt{\frac{20}{\pi}}$, we have that

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}r^2}\right|_{r=\sqrt{\frac{20}{\pi}}} = -6\pi\sqrt{\frac{20}{\pi}} < 0$$

and hence since $V''(r) < 0$ at $r = \sqrt{\frac{20}{\pi}}$, this value of r gives a maximum value of V.

(d) From before, we have that

$$h = \frac{120 - 2\pi r^2}{2\pi r}$$

Hence when $r = \sqrt{\frac{20}{\pi}}$, we have

$$h = \frac{120 - 2\pi\left(\sqrt{\frac{20}{\pi}}\right)^2}{2\pi\sqrt{\frac{20}{\pi}}} = \frac{40}{\pi}\sqrt{\frac{\pi}{20}}$$

Hence

$$\frac{h}{r} = \frac{40}{\pi}\sqrt{\frac{\pi}{20}} \times \sqrt{\frac{\pi}{20}} = 2$$

Answer: $\frac{h}{r} = 2$

Question Sheet: **Sheet 4**

Model Solution No: 6

(a) (i) **Answer:** $T - 3.2 = 2a$ (ii) **Answer:** $4g - T = 4a$ (or $39.2 - T = 4a$)

(b) Solving the equations from part (a) (i) and (a) (ii) simultaneously gets you the results.

Answer: $a = 6$ m s^{-2}

(c) Use $s = ut + \frac{1}{2}at^2$ with $s = 2$, $u = 0$ and $a = 6$ to get

$$t = \sqrt{\frac{2s}{a}} = 0.8164...$$

Answer: 0.82 s (2 sf)

(d) **Answer:** e.g. Q has constant acceleration during its motion

Question Sheet: **Sheet 4**

Model Solution No: 7

(a) The sum of probabilities must equal 1. Thus:

$$\frac{k}{1(1+1)} + \frac{k}{2(2+1)} + \frac{k}{3(3+1)} + \frac{k}{4(4+1)} = 1$$

This gives $k = \frac{5}{4}$

Answer: $k = \frac{5}{4}$

[Tip: use your value of k and check that, indeed, the probabilities sum to 1.]

(b) $P(-2 \leq X \leq 2.5) = P(X = 1) + P(X = 2) = \dfrac{5/4}{1(1+1)} + \dfrac{5/4}{2(2+1)} = \dfrac{5}{6}$

Answer: $\frac{5}{6}$

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