CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 3**

Model Solution No: 1

(a) Use the substitution $u = \sqrt{x}$ to obtain

$$5u^2 - 12u + 4 = 0$$

which factorises to $(5u-2)(u-2) = 0$.

So the values of u are $u = 2$ or $u = \frac{2}{5}$.

Then we have to 'undo' our substitution, so $\sqrt{x} = 2$, $\sqrt{x} = \frac{2}{5}$. Squaring both sides then gives the result.

Answer: $x = 4$, $x = \frac{4}{25}$

(b) This is a disguised quadratic again. You notice this because we have a 2^{2y} which is the square of 2^y. This suggests a substitution of $w = 2^y$.

Note if $w = 2^y$, then $2^{2y+2} = 2^{2y} \times 2^2 = 4w^2$. Hence our equation becomes

$$4w^2 - 35w + 24 = 0$$

which factorises to $(w-8)(4w-3) = 0$.

So the values of w are $w = 8$ and $w = \frac{3}{4}$.

Again, we have to 'undo' our substitutions. So if $w = 8$, then $2^y = 8$, so $y = 3$ by inspection.

Alternatively if $w = \frac{3}{4}$, then $2^y = \frac{3}{4}$. We can find this solution by taking logs to get $y = \frac{\log(3/4)}{\log 2}$.

Answer: $y = 3$ or $y = -0.415$ (3 sf). [Tip: substitute your values back into the equation to check they work.]

(c) Using $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\begin{aligned}
&5\cos^2\theta - \sin\theta - 1 = 0 \\
&\Rightarrow 5(1 - \sin^2\theta) - \sin\theta - 1 = 0 \\
&\Rightarrow 5 - 5\sin^2\theta - \sin\theta - 1 = 0 \\
&\Rightarrow 5\sin^2\theta + \sin\theta - 4 = 0 \\
&\Rightarrow (5\sin\theta - 4)(\sin\theta + 1) = 0 \\
&\Rightarrow \sin\theta = \frac{4}{5} \ \text{ or } \ \sin\theta = -1
\end{aligned}$$

If $\sin\theta = \frac{4}{5}$, then the two values of θ in range are 53.1 and 126.9. [You can obtain this by using your favourite method for solving trig. equations.]

If $\sin\theta = -1$, then $\theta = -270$ by inspection.

Answer: $\theta = 53.1,\ 126.9,\ 270$ (given to 3 sf where relevant)

Question Sheet: **Sheet 3**

Model Solution No: 2

(a) (i) **Solution:** It is easier to show this by expanding the RHS:

$$\begin{aligned}(a-b)(a^2+ab+b^2) &\equiv a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &\equiv a^3-b^3\end{aligned}$$

as required.

(a) (ii) **Solution:** Again, start by expanding the RHS:

$$\begin{aligned}a^2+b^2+(a+b)^2 &\equiv a^2+b^2+a^2+2ab+b^2\\ &\equiv 2a^2+2ab+2b^2\\ &\equiv 2(a^2+ab+b^2)\end{aligned}$$

as required.

(b) **Solution:** From part (a) (ii), we have that

$$a^2+ab+b^2=\frac{a^2+b^2+(a+b)^2}{2}$$

Since a^2, b^2 and $(a+b)^2$ are each non-negative for all a, b, so is their sum.

Thus $a^2+ab+b^2 \geq 0$ for all a, b as required.

(c) **Solution:**

$$\begin{aligned}p>q &\Rightarrow p-q>0\\ &\Rightarrow (p-q)(p^2+pq+q^2)>0 \qquad (*)\\ &\Rightarrow p^3-q^3>0\\ &\Rightarrow p^3>q^3\end{aligned}$$

as required.

In line $(*)$, we used part (a) (i) by multiplying both sides by (p^2+pq+q^2). We also used part (a) (ii) to preserve the direction of the inequality.

Question Sheet: **Sheet 3**

Model Solution No: 3

(a) **Solution:** Since the centre is $(5, 2)$ and the radius is $\sqrt{17}$, we can write down the equation of C as:

$$(x-5)^2 + (y-2)^2 = 17$$

Expanding then gives $x^2 - 10x + 25 + y^2 - 4y + 4 = 17 \Rightarrow x^2 - 10x + y^2 - 4y + 12 = 0$, as required.

(b) The equation of the lines are of the form $y = mx - 1$. This intersects the circle at the points whose x coordinates satisfy:

$$\begin{aligned}
&x^2 - 10x + (mx-1)^2 - 4(mx-1) + 12 = 0 \\
&\Rightarrow x^2 - 10x + m^2x^2 - 2mx + 1 - 4mx + 4 + 12 = 0 \\
&\Rightarrow (1+m^2)x^2 + (-6m-10)x + 17 = 0
\end{aligned}$$

For the lines to be tangents, this equation can only have one solution (as the circle and lines intersect once only). So the discriminant must equal 0. This gives

$$\begin{aligned}
&(6m+10)^2 - 4(1+m^2)(17) = 0 \\
&\Rightarrow 36m^2 + 120m + 100 - 68 - 68m^2 = 0 \\
&\Rightarrow 32m^2 - 120m - 32 = 0 \\
&\Rightarrow 4m^2 - 15m - 4 = 0 \\
&\Rightarrow (4m+1)(m-4) = 0
\end{aligned}$$

So $m = 4$ or $m = -\frac{1}{4}$. From the diagram, clearly l_1 is the one with (positive) gradient $m = 4$ and l_2 is the one with (negative) gradient $-\frac{1}{4}$. Thus:

Answer: Equation of l_1 is $y = 4x - 1$ Equation of l_2 is $y = -\frac{1}{4}x - 1$.

Equivalent answers are acceptable.

CRASHMATHS
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Question Sheet: **Sheet 3**

Model Solution No: 4

(a) **Solution:** By the factor theorem, if $(x+2)$ is a factor of $f(x)$, then $f(-2) = 0$. Using this gives:

$$\begin{aligned} &f(-2) = a(-2)^3 + b(-2)^2 + 5(-2) - 6 = 0 \\ &\Rightarrow -8a + 4b - 10 - 6 = 0 \\ &\Rightarrow 4b - 8a = 16 \\ &\Rightarrow b - 2a = 4 \end{aligned}$$

as required.

(b) (i) Again, by the factor theorem, if $(3-x)$ is a factor of $f(x)$, then $f(3) = 0$. Using this, you can obtain the result.

Answer: $3a + b = -1$

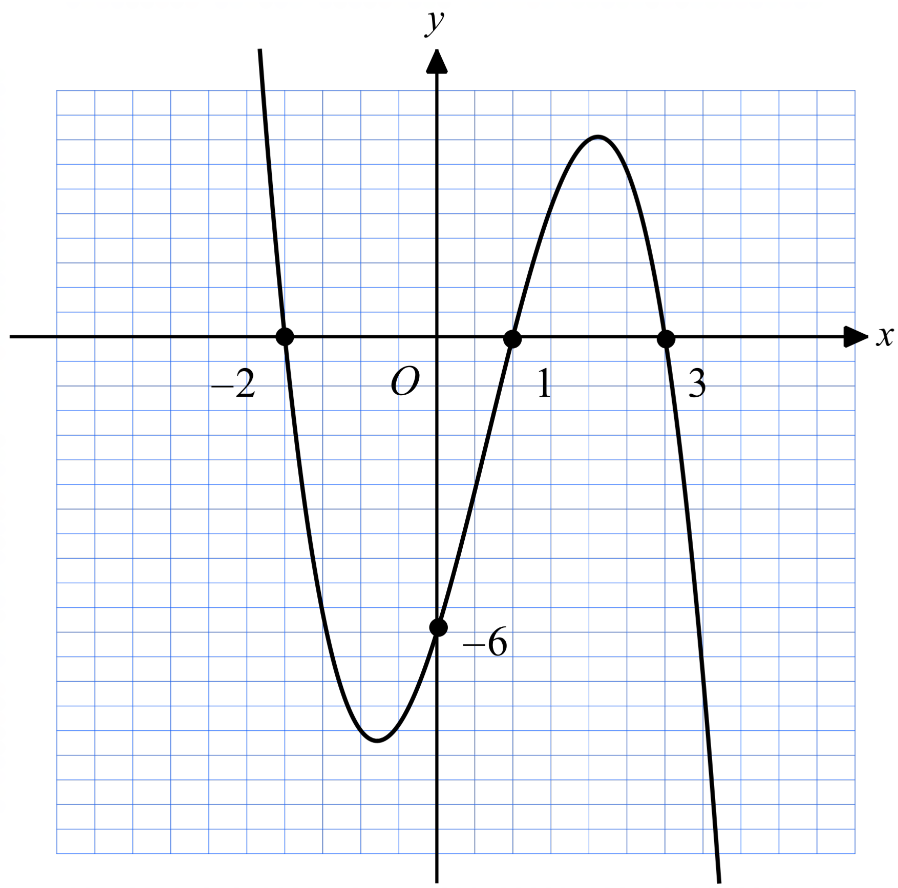
(b) (ii) Solve $3a + b = -1$ and $b - 2a = 4$ simultaneously to get the result.

(c) You have $f(x) = -x^3 + 2x^2 + 5x - 6$ and we know that it has factors $(3-x)$ and $(x+2)$. There are now a few ways to find the third factor. The quickest is inspection: this gives that the other factor has to be $(x-1)$.

Why? We need it to be x to make the $-x^3$ when we multiply out and we need the -1 there so when we multiply out, we also get the constant 6. Alternatively, you can use a long division approach.

Answer: $f(x) = (3-x)(x+2)(x-1)$.

(d)



Question Sheet: **Sheet 3**

Model Solution No: 5

(a) $N = +400t - 450t + 8000$

Answer: $N = -50t + 8000$

(b) **Answer:** e.g. the infected cells are being destroyed at a faster rate than they are growing / the gradient of N against t is negative so N decreases with time

(c) $0 = -50t + 8000 \Rightarrow t = \frac{8000}{50} = 160$

Answer: 160 days

(d) **Answer:** not valid for $t > 160$ / eventually predicts a negative number of infected cells / assumes patient is going under constant/continuous treatment

Question Sheet: **Sheet 3**

Model Solution No: 6

(a) We integrate a wrt t to obtain an expression for the velocity, so

$$v = \int (3t^2 - 4)\ \mathrm{d}t = t^3 - 4t + c$$

At time $t = 0, v = 0$, so $c = 0$.

Answer: $v = t^3 - 4t$

(b) We seek the area under the velocity-time graph between $[0, 5]$. This is given by

$$\text{Area} = -\int_0^2 (t^3 - 4t)\ \mathrm{d}t + \int_2^5 (t^3 - 4t)\ \mathrm{d}t$$

We have partitioned the integral because for $0 < x < 2$, the curve is below the x-axis, while for $2 < x < 5$, it is above the x-axis.

We calculate each integral separately,

$$-\int_0^2 (t^3 - 4t)\ \mathrm{d}t = -\left[\frac{1}{4}t^4 - 2t^2\right]_0^2$$

and after substituting in the limits, this evaluates to 4. The other integral can be calculated similarly (the answer should be 441/4)

Adding together the results gives the final answer of $\frac{457}{4}$. [Left as an exercise to fill in the details.]

Answer: $\dfrac{457}{4}$ m

Question Sheet: **Sheet 3**

Model Solution No: 7

Solution:

(a) **Answer:** Simple random sampling

(b) **Answer:** list of random numbers may contain repeats

(c) **Answer:** 'tr' indicates an amount of rainfall between 0 and 0.05 mm - 0.025 selected because it is the average of these values

(d) Mean given by $\frac{\sum r}{n} = \frac{102.95}{20} = 5.1475$ mm

Standard deviation given by $\sqrt{\frac{\sum r^2}{n} - \bar{x}^2} = \sqrt{\frac{3119}{20} - 5.1475^2} = 11.37...$ mm

Answer: Mean is 5.15 mm, standard deviation is 11.4 mm.

(e) **Answer:** e.g. Beijing's rainfall data contains outliers/is skewed and this will affect the mean and standard deviation

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