A Level
Maths

Bronze Set C, Paper 1

<table>
<tr><th>Question</th><th>Solution</th><th>Partial Marks</th><th>Guidance</th></tr>
<tr><td>1</td><td>$4x^3 - 8xy - 4x^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$ *
$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 2xy}{x^2 - y^3}$

So at (2, 3), $\frac{dy}{dx} = \frac{2^3 - 2(2)(3)}{2^2 - 3^3} = \frac{4}{23}$

So eq of tangent at (2, 3) is
$y - 3 = \frac{4}{23}(x - 2)$
$\Rightarrow 23y - 69 = 4x - 8$
$\Rightarrow 4x - 23y + 61 = 0$</td><td>B1
M1

A1

M1*

M1(dep*)

A1
[6]</td><td>Correct differentiation of the $4x^2y$ term (underlined)
Attempts derivative of y^4 term obtaining expression of the form $Ay^3\, dy/dx$, $A \neq 0$, (double underlined)
Fully correct differentiation. Do not necessarily need to rearrange for $dy/dx = \ldots$. If they obtain (*) (with no spurious terms), it is A1

Substitutes (2, 3) into their dy/dx to obtain value for gradient. May do this once they reach the stage (*) and **then** re-arrange for dy/dx which is OK
This is not formally dependent on the 1st M1 but does requires them to have at least two y terms in their derivative coming from y^4 and $4x^2y$
Writes down equation of the tangent in any form using their gradient. Or if using $y = mx + c$, need to see them employ a complete method to find c
Correct equation of the tangent in the correct form (allow non-zero integer multiples)</td></tr>
<tr><td>2</td><td><table><tr><th>n</th><th>$1^3 + 2^3 + \ldots + n^3$</th><th>$(1 + 2 + \ldots + n)^2$</th><th>Equality</th></tr><tr><td>1</td><td>1</td><td>1</td><td>yes</td></tr><tr><td>2</td><td>9</td><td>9</td><td>yes</td></tr><tr><td>3</td><td>36</td><td>36</td><td>yes</td></tr><tr><td>4</td><td>100</td><td>100</td><td>yes</td></tr><tr><td>5</td><td>225</td><td>225</td><td>yes</td></tr><tr><td>6</td><td>441</td><td>21</td><td>yes</td></tr></table>
Hence $1^3 + 2^3 + \ldots + n^3 \equiv (1 + 2 + \ldots + n)^2$ for all positive integers from 1 to 6 inclusive</td><td>M1

M1

A1

[3]</td><td>**Need not see a table**
Shows equality for at least one positive integer from 1 to 6 inclusive
Complete method to show the equality for $n = 1, 2, 3, 4, 5, 6$

Complete and convincing proof. Need to show equality (or ‘truth’) for all the required cases and then give a final conclusion at the end
SC: completely correct proof by exhaustion with conclusion excluding the cases $n = 1$ and $n = 6$ can score SCB2</td></tr>
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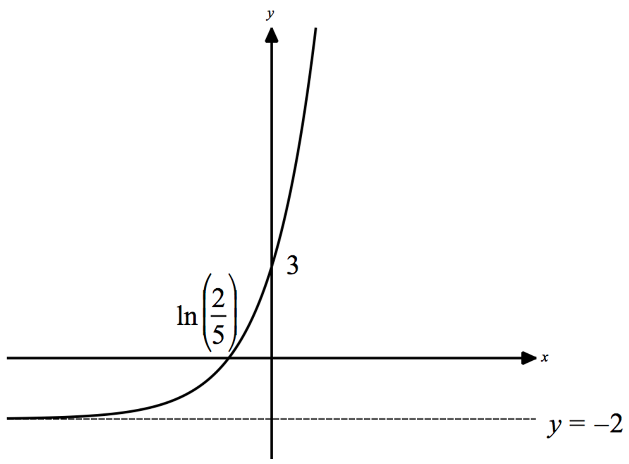
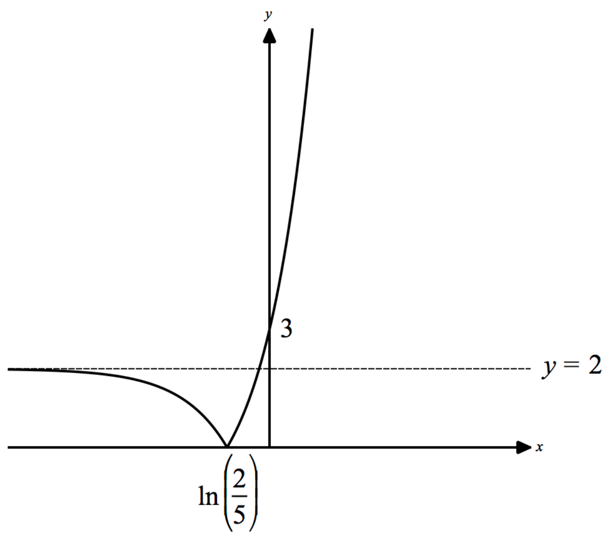
3	$y=\int\left(\frac{2-x^2}{x}\right)dx$ $=\int\left(\frac{2}{x}-x\right)dx$ $=2\ln x-\frac{1}{2}x^2+c$ When $x = 1, y = 4$, so $4=-\frac{1}{2}+c\Rightarrow c=\frac{9}{2}$ Hence $y=2\ln x-\frac{1}{2}x^2+\frac{9}{2}$	M1* M1 A1 oe M1(dep*) A1 **[5]**	States or implies intention to integrate RHS (no need to select a method to integrate here) Obtains integral of the form $A\ln(x) + bx^2 + c$ where $a, b \neq 0$ Obtains correct integral including constant oe Allow $\ln(x^2)$ and $2\ln(\|x\|)$ (though $\|x\|$ is redundant here as $x > 0$) Uses initial conditions to find the value of their constant Obtains correct expression for y in terms of x Allow $\ln(x^2)$ and $2\ln(\|x\|)$
4 (a)	$g(x)=(4+3x)^{-\frac{1}{2}}$ $=(4)^{-\frac{1}{2}}\left(1+\frac{3}{4}x\right)^{-\frac{1}{2}}$ $=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(\frac{3}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{3}{4}x\right)^2+\ldots\right)$ $=\frac{1}{2}\left(1-\frac{3}{8}x+\frac{27}{128}x^2+\ldots\right)$ $=\frac{1}{2}-\frac{3}{16}x+\frac{27}{256}x^2+\ldots$	 B1 M1 A1 A1 A1 **[5]**	 $4^{-1/2}$ or ½ outside the brackets or appearing as constant term in candidate's expansion Expands $(1 + px)^{-1/2}$ obtain 2 out of the 3 required terms, $p \neq 1$, unsimplified or better Expands $(1 + px)^{-1/2}$ correctly obtaining first 3 terms, $p \neq 1$, unsimplified or better. Note for M1 A1 : need to use a <u>consistent</u> p on the RHS, but not necessarily on the LHS Correct constant **and** linear term (allow $-0.1875x$) Correct quadratic term (allow $0.10546875x^2$). Cao
4 (b)	$\frac{1}{\sqrt{4+3\left(\frac{1}{3}\right)}}=\frac{1}{\sqrt{5}}$ $\quad=\frac{1}{5}\sqrt{5}$ (so $k = 1/5$)	B1 **[1]**	Cao Allow just value of k stated

4 (c)	$\frac{1}{5}\sqrt{5}=\frac{1}{2}-\frac{3}{16}\left(\frac{1}{3}\right)+\frac{27}{256}\left(\frac{1}{3}\right)^2$ $\Rightarrow \frac{1}{5}\sqrt{5}=\frac{115}{256}$ $\Rightarrow \sqrt{5}=\frac{575}{256}$	M1 A1 cao **[2]**	Substitutes $x=\frac{1}{3}$ into their (a) **and** equates this to their $k\sqrt{5}$ M1 can be implied for $\sqrt{5}=\frac{1}{k}\left(\text{their } \frac{115}{256}\right)$ with their k from (b) Obtains correct approximation as a fraction Allow use of approximation notation throughout
5 (a)	$(x-1)^2-1+(y-5)^2-5^2-8=0$ $\Rightarrow (x-1)^2+(y-5)^2=34$ So centre is (1, 5) Radius is $\sqrt{34}$	M1 A1 A1ft A1ft **[4]**	Correct method to complete the square on the x terms or the y terms So e.g. need to see $(x-1)^2-1^2$ or $(y-5)^2-5^2$ oe Obtains $(x-1)^2+(y-5)^2=34$ (or constant on LHS) Correct centre ft their LHS Correct radius ft their RHS
5 (b)	X coordinates are 3 units either side of the centre So x coordinate of A is –2 and x coordinate of B is 4 $\sqrt{34-9}=\sqrt{25}=5$, so chord is 5 units above the centre Hence A = (–2, 10), B = (4, 10)	 B1 M1 A1 **[3]**	 States correct x coordinates at any stage (allow labels switched) Uses Pythagoras to find vertical distance of centre from chord using their (a) Correct y coordinate of A and B

6 (a) (i)		B1 B1 B1 **[3]**	Correct shape – an exponential (growth) shaped curve drawn in any position For correct x and y intersections. <u>May see these in script body</u> Allow in coordinate form and condone coordinates given the wrong way around if the point is marked on the correct axis Equation of asymptote given as $y = -2$. Note that this asymptote does not have to appear drawn on the curve **but it must appear that the curve has an asymptote at $y = -2$**. Note it is not enough to have asymptote drawn with –2 indicated on the y axis – <u>we must see the equation</u>
6 (a) (ii)		B1ft B1ft B1ft **[3]**	Correct shape – for the correct shape including the cusp. The curve to the left of the cusp must appear to have the correct curvature. For the follow through their curve in (a/i) must have appeared above and below the x axis For correct x and y intersections. <u>May see these in script body</u> Allow in coordinate form and condone coordinates given the wrong way around if the point is marked on the correct axis. Ft their (a/i) Equation of asymptote given as $y = 2$. Note that this asymptote does not have to appear drawn on the curve **but it must appear that the curve has an asymptote at $y = 2$**. SC: do not penalise this if the B0 scored for wrong curvature on left of cusp

6 (b)	$5e^x - 2 = 1 \Rightarrow x = \ln\frac{3}{5}$ Or $5e^x - 2 = -1 \Rightarrow x = \ln\frac{1}{5}$	B1 M1 A1 **[3]**	Correct exact value of x Obtains correct equation satisfied by second value of x Correct second exact value of x Notes on squaring: they can square to obtain $(5e^x - 2)^2 = 1$, but then need to employ a complete method to find $e^x = \ldots$, $e^x = \ldots$
6 (c)	Values of k are $\{k \in \mathbb{R} : k \geq 2\} \cup \{k = 0\}$	B1 **[1]**	Correct set of values of k Allow omission of $\in \mathbb{R}$ Allow equivalent sets e.g. $\{k \in \mathbb{R} : k \geq 2 \;\cup\; k = 0\}$
7 (a)	$x_1 = -1$, $x_2 = 1$, $x_3 = -1$, $x_4 = 1$, etc. so sequence is periodic with order 2	B1 B1 **[1]**	Illustrates the series is periodic by writing out at least the first four terms and conclusion, e.g. 'hence periodic', 'as required', Allow e.g. 'if n is odd, the term is –1, if n is even, the term is 1, (so the terms oscillate between –1 and 1). Hence periodic' (or better) States correct order of the sequence
7 (b) (i)	If k is odd, the sum is –1	B1 **[1]**	Cao
7 (b) (ii)	If k is even, the sum is 0	B1 **[1]**	Cao

8 (a)	$\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}\equiv\dfrac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$ $\equiv\dfrac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}$ $\equiv\dfrac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}$ $\equiv\dfrac{2}{\sin\theta}$ $\equiv 2\operatorname{cosec}\theta$ **AG**	M1* M1**(dep*) A1 A1 **[4]**	Attempts a common denominator Expands brackets and uses $\sin^2\theta + \cos^2\theta = 1$ Correct workings up to this stage Shows the result convincingly with no errors seen
6 (c)	Eq equivalent to $2\operatorname{cosec}3\theta=\sqrt{5}\Rightarrow\sin 3\theta=\dfrac{2}{\sqrt{5}}$ Principal value of $3\theta=\sin^{-1}\left(\dfrac{2}{\sqrt{5}}\right)=1.1071...$ Other values in range are: $\pi-1.1071...=2.0344...$ and $1.1071...+2\pi=7.3903...$ and $2.0344...+2\pi=8.3176...$ Hence $3\theta=1.1071,\ 2.0344,\ 7.3903,\ 8.3176$ $\Rightarrow\theta=0.37,\ 0.68,\ 2.46,\ 2.77$	M1 M1 A1 M1 A1 cao **[5]**	Writes equation correctly in terms of cosec3θ and attempts to re-arrange for sin(3θ) Attempts to find the principal value of 3θ using $\sin^{-1}$(their 2/√5). Their equation should be well-defined Obtains at least two correct values of 3θ in range (this can include the correct principal value) Method to find all four values of 3θ in range and divides them each by 3 to find the values of θ Obtains correct values of θ to 2dp. Cao
9 (a)	110	B1 **[1]**	Cao
9 (b)	$\dfrac{dp}{dt}=0.14t+4$ And since $\dfrac{dp}{dt}>0$ for all $t\geq 0$, the size of the population increases with time	M1 A1 **[2]**	Obtains correct dp/dt States $dp/dt > 0$ for all time $t \geq 0$ and concludes Note 1: allow ≥ 0 for dp/dt Note 2: do not allow $dp/dt > 0$ for all time, but DO allow 'for all time in the model'

9 (b) ALT	**Alt 1:** $0.07t^2+4t+110=0.07\left(t^2+\frac{400}{7}t\right)+110$ $=0.07\left(t+\frac{200}{7}\right)^2-0.07\left(\frac{200}{7}\right)^2+110$ $=0.07\left(t+\frac{200}{7}\right)^2+\frac{370}{7}$ Minimum point of the curve is at $t = -200/7$ and since curve is U shaped, the size of the population increases with time since t in the model is only for $t \geq 0$ (or t in model $> -200/7$)	M1 A1 **[2]**	Correctly completes the square (or correctly obtains value of t at which minimum occurs) ***OR* ALT 2:** computes discriminant = –14.8 Explains why the population increases with time with reference to domain of t in model (similar reasoning if using discriminant)
9 (c)	$110=\frac{440a}{1+a}$ $\Rightarrow 110(1+a)=440a$ $\Rightarrow 110+110a=440a$ $\Rightarrow 330a=110$ $\Rightarrow a=\frac{1}{3}$ **AG**	M1* M1(dep*) A1 **[3]**	Sets their $110 = 440a/(1 + a)$ Solves for a Shows that $a = 1/3$ convincingly with no errors seen SCB1 only for a 'backward approach', i.e. using $a = 1/3$, finding that Model 2 gives $p(0) = 110$ and this agrees with Model 1
9 (d)	$330=\frac{440e^{0.1t}}{3+e^{0.1t}}$ $\Rightarrow 990+330e^{0.1t}=440e^{0.1t}$ $\Rightarrow e^{0.1t}=9$ $\Rightarrow 0.1t=\ln 9$ $\Rightarrow t=\frac{1}{0.1}\ln 9=21.9...$ So about 22 weeks	M1* M1(dep*) A1 **[3]**	Equates model 2 with $a = 1/3$ to 330 and solves for $e^{0.1t}$ Takes natural logs on both sides with use of $\ln(e) = 1$ seen Correct time. Awrt 22. No need to add on 'weeks' at the end, but A0 if the wrong units given

9 (e)	In Model 1, p grows without bound / $p \to \infty$ as $t \to \infty$ In Model 2, (p saturates with) $p \to 440$ as $t \to \infty$ So Model 2 is best because the population cannot grow without bound / the population size is limited by resources	B1* B1* B1(dep*) **[3]**	Correct description of the long term behaviour of model 1 Correct description of long term behaviour of model 2 Suggests that Model 2 is better and gives a suitable reason in context. Just need a reason about why $p \to \infty$ is not realistic ora
10 (a)	When $y = \frac{\pi}{9}$, $x = 4\tan\left(3\frac{\pi}{9}\right) = 4\tan\frac{\pi}{3} = 4\sqrt{3}$ Hence P lies on the curve	B1 **[1]**	Shows P lies on the curve <u>ALT: note if they sub in x and use arctan to find y, they must use its domain to justify their choice of ‘θ’</u>
10 (b)	$\frac{dx}{dy} = 12\sec^2 3y$ $\Rightarrow \frac{dy}{dx} = \frac{1}{12\sec^2 3y}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{12(1+\tan^2 3y)} = \frac{1}{12+12\tan^2 3y}$ Now, $\tan(3y) = x/4 \Rightarrow \frac{dy}{dx} = \frac{1}{12+12\left(\frac{x}{4}\right)^2}$ Hence (multiplying by 4 gives) $\frac{dy}{dx} = \frac{4}{48+3x^2}$ **AG**	B1 M1 M1 A1 **[4]**	Correct dx/dy For use of $dy/dx = 1/(dx/dy)$ Use of $\sec^2(3y) = 1 + \tan^2(3y)$ **and** attempts to replace $\tan^2(3y)$ with Convincing proof of the given result with no errors
10 (c)	at P, $\frac{dy}{dx} = \frac{1}{12+3(4\sqrt{3})^2} = \frac{4}{48+3(4\sqrt{3})^2} = \frac{1}{48}$ so gradient of the normal is –48 Equation of normal is then $y - \frac{\pi}{9} = -48\left(x - 4\sqrt{3}\right)$ $\Rightarrow y = -48x + 192\sqrt{3} + \frac{\pi}{9}$	M1 A1ft M1 A1 cao **[4]**	Substitutes $x = 4\sqrt{3}$ into dy/dx OR dx/dy Uses their value for the gradient at $x = 4\sqrt{3}$ to write down the gradient of the normal Uses their gradient of the normal with the coordinates of P to write down the equation of the normal in any form Obtains correct equation of the normal in the required form Exact values for m and c only

11 (a)	$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC}$ $= (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + (2\mathbf{i} - \mathbf{j} + \mathbf{k})$ $= 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ Unit vector is then $\widehat{OB} = \frac{1}{\sqrt{3^2 + 2^2 + 3^2}}(3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ $= \frac{1}{\sqrt{22}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$	B1 M1 A1 oe **[3]**	Correct direction vector $\overrightarrow{OB}$ Correct method to find the unit vector ft their $\overrightarrow{OB}$ Correct unit vector oe

11 (b) 'Geometric'	Let P_1 be the midpoint of OB and P_2 be the midpoint of AC $\frac{1}{2}\overrightarrow{OB}=\overrightarrow{OP_1}$ Also have $\overrightarrow{OP_2}=\overrightarrow{OC}+\frac{1}{2}\overrightarrow{CA}$ $=\frac{1}{2}\overrightarrow{OC}+\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{CA}$ (using $\overrightarrow{OC}=\frac{1}{2}\overrightarrow{OC}+\frac{1}{2}\overrightarrow{AB}$) $=\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{AB}+\overrightarrow{CA})$ $=\frac{1}{2}\overrightarrow{OB}=\overrightarrow{OP_1}$ Hence $\overrightarrow{OP_1}=\overrightarrow{OP_2}$ and the lines bisect each other	M1 M1 A1 A1 **[4]**	**There are multiple methods to this question. We only give two ways and their mark schemes. Mark others similarly** Shows intention to find position vector of midpoint of AC Attempts to associate position vector of midpoint of AC with the midpoint of the line BC Obtains $\overrightarrow{OP_2}=\frac{1}{2}\overrightarrow{OB}$ or better Shows the result convincing with explanation of the result
11 (b) ALT 'Computational'	$\frac{1}{2}\overrightarrow{OB}=\frac{1}{2}(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ $\overrightarrow{AC}=(2\mathbf{i}-\mathbf{j}+\mathbf{k})-(\mathbf{i}+3\mathbf{j}+2\mathbf{k})=(\mathbf{i}-4\mathbf{j}-\mathbf{k})$ Then $\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}=(\mathbf{i}+4\mathbf{j}+2\mathbf{k})+\frac{1}{2}(\mathbf{i}-4\mathbf{j}-\mathbf{k})$ $=\frac{1}{2}(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ Since $\frac{1}{2}\overrightarrow{OB}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}$, the diagonals of the parallelogram bisect each other	M1* A1 M1(dep*) A1 **[4]**	Attempts to find a vector $\overrightarrow{AC}$ Correct vector $\overrightarrow{AC}$ Uses a complete method to show the lines bisect each other Shows the result convincingly with conclusion

11 (c)	$\lvert OP\rvert = \frac{1}{2}\lvert OB\rvert = \frac{1}{2}\sqrt{3^2+2^2+3^2} = \frac{1}{2}\sqrt{22}$ $\lvert PC\rvert = \frac{1}{2}\lvert\overrightarrow{AC}\rvert = \frac{1}{2}\sqrt{1^2+4^2+1^2} = \frac{3}{2}\sqrt{2}$ $\lvert\overrightarrow{OC}\rvert = \sqrt{2^2+1^2+1^2} = \sqrt{6}$ So by the cosine rule $\cos(\angle OPC) = \dfrac{\left(\frac{1}{2}\sqrt{22}\right)^2 + \left(\frac{3}{2}\sqrt{2}\right)^2 - \left(\sqrt{6}\right)^2}{2\left(\frac{1}{2}\sqrt{22}\right)\left(\frac{3}{2}\sqrt{2}\right)}$ $\Rightarrow \cos(\angle OPC) = \dfrac{4\sqrt{11}}{33}$ $\Rightarrow \angle OPC = 66.29...^\circ$	M1 A1 M1 A1 **[4]**	Employs a correct method to find the length of each side Correct lengths of each side of the triangle Uses the correct formula for the cosine rule with their side lengths leading to a value for the angle Obtains the correct angle. Awrt 66º (or 1.2 radians)
11 (c) ALT	$\cos(\angle OPC) = \dfrac{\overrightarrow{OB}\cdot\overrightarrow{CA}}{\lvert\overrightarrow{OB}\rvert\lvert\overrightarrow{CA}\rvert}$ $= \dfrac{(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})\bullet(-\mathbf{i}+4\mathbf{j}+\mathbf{k})}{\sqrt{3^2+2^2+3^2}\sqrt{1^2+4^2+1^2}}$ $= \dfrac{-3+8+3}{\sqrt{22}\sqrt{18}}$ So $\Rightarrow \cos(\angle OPC) = \dfrac{4\sqrt{11}}{33}$ $\Rightarrow \angle OPC = 66.29...^\circ$	M1 M1 A1 A1 **[4]**	Writes down correct expression for the angle Employs correct method to find dot product of their two vectors and the lengths of these vectors Correct expression for the cosine of the angle Obtains correct angle

12 (a)	Assume for a contradiction that the square root of 6 is rational, i.e. $\sqrt{6}=\frac{a}{b}$, $a,b\in\mathbb{Z},b\neq 0$ and a, b coprime Then $6=\frac{a^2}{b^2}\Rightarrow a^2=6b^2$ $\Rightarrow a^2$ is a multiple of 6 $\Rightarrow a$ is a multiple of 6, so $a = 6k$ for some $k\in\mathbb{Z}$ Then $(6k)^2=6b^2\Rightarrow b^2=6k^2$ $\Rightarrow b^2$ is a multiple of 6 $\Rightarrow b$ is a multiple of 6 However this is a contradiction because a and b were taken to be coprime Hence $\sqrt{6}$ is irrational	M1 A1 M1(dep*) A1 **[4]**	Uses the method of proof by contradiction by writing $\sqrt{6}=\frac{a}{b}$ with a and b correctly defined Allow hcf(a, b) = 1 or gcd(a, b) = 1 instead of 'coprime' Shows that a is a multiple of 6 using a logical argument Uses the result that a is a multiple of 6 to show that b is a multiple of 6 Complete and convincing proof showing logically that $\sqrt{6}$ is irrational. They must explain what the contradiction is and conclude 'hence irrational'. All 3 previous marks are necessary here.
12 (b)	She has assumed that the sum of any two irrational numbers is irrational which is not necessarily true For example, take $a = \sqrt{2}$ and $b = 1 - \sqrt{2}$ which are both irrational. Then $a + b = 1$ which is rational/not irrational Hence Irini's reasoning is incorrect	B1 B1 **[2]**	States the assumption she has made However/but etc. in place of 'for example' counts as a blue element Uses a counter-example to disprove her assumption + at least one of the blue elements (or similar) to form a coherent argument SC: if B0 B0 scored but a suitable counter example is used to disprove the assumption (i.e. assumption not explicitly stated), allow SCB1
12 (c)	(This is a contradiction because) $\frac{r^2-5}{2}$ is rational but $\sqrt{6}$ is not / irrational	B1 **[1]**	Explains the contradiction

13	This scheme is split into sections. The stage 2 section has 3 alternatives, so consider these as you mark **Do not forget the process mark →**	M1 [1 mark max.]	This is a strategy/process mark. We are looking for a complete method. Need to see the candidate do the following (in any order): 1) attempt to find the coordinates of P using differentiation 2) attempt to find the integral $\int_0^a x\sqrt{4-x^2}\,dx$, $a \neq 0$, $0 < a < 2$ 3) use their value for the integral and the coordinates of P to find the area of R
Stage 1: finding *P*	$\frac{dy}{dx} = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$ At P, $\frac{dy}{dx} = 0 \Rightarrow \sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}}$ $\Rightarrow 4 - x^2 = x^2$ $\Rightarrow x^2 = 2$ $\Rightarrow x = \sqrt{2}$ So $P = (\sqrt{2}, 2)$	M1 M1 A1 [3 marks max.]	Attempts to find the derivative using the product rule. Need to see an attempt of the chain rule to tackle the $\sqrt{(4-x^2)}$ Sets their derivative = 0 and solves for x (their derivative must yield a 3TQ) Obtains correct x coordinate of P NB: y coordinate of P not needed
Stage 2: finding $\int_0^a x\sqrt{4-x^2}\,dx$	$\int_0^{\sqrt{2}} x\sqrt{4-x^2}\,dx = \left[\frac{2}{3}(4-x^2)^{\frac{3}{2}} \times -\frac{1}{2}\right]_0^{\sqrt{2}}$ $= \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}}$ $= -\frac{1}{3}(4-2)^{\frac{3}{2}} + \frac{1}{3}(4-0)^{\frac{3}{2}}$ $= \frac{8}{3} - \frac{2}{3}\sqrt{2}$	M1* A1 M1(dep*) A1 [4 marks max.]	Uses the reverse chain rule obtaining integral of the form $A(4-x^2)^{3/2}$, $A \neq 0$ Obtains correct integral (ignore limits) Substitutes **correct** limits (ft their P) **in the correct order** Obtains correct value for the integral For this mark, allow a non-exact value (awrt 1.72)

Stage 3: finding area of *R*	Area of the triangle is $\frac{1}{2}(\sqrt{2})(2)=\sqrt{2}$ So area of *R* $=\frac{8}{3}-\frac{2}{3}\sqrt{2}-\sqrt{2}$ $=\frac{8}{3}-\frac{5}{3}\sqrt{2}$	M1 A1 cao [2 marks max.] **[10]**	Complete method to find the area of *R* using their value for the integral and their area of the triangle Correct exact area of *R*, final answer
Stage 2 ALT 1	Use $u=4-x^2$ to obtain: $\int_0^{\sqrt{2}} x\sqrt{4-x^2}\,dx=\int_4^2 \not{x}\sqrt{u}\left(-\frac{1}{2\not{x}}du\right)$ $=-\frac{1}{2}\int_4^2\sqrt{u}du$ $=\left[-\frac{u^{\frac{3}{2}}}{3}\right]_4^2$ $=-\frac{2^{\frac{3}{2}}}{3}+\frac{4^{\frac{3}{2}}}{3}$ $=\frac{8}{3}-\frac{2}{3}\sqrt{2}$	M1* A1 M1(dep*) A1 [4 marks max.]	Chooses a suitable substitution, attempts to change the area element and obtains integrand in their new variable Obtains correct integrand (ignore limits) Obtains integral of the form $Bu^{3/2}$, $B \neq 0$, and substitutes **correct** limits (ft their *P*) **in the correct order** Obtains correct value for the integral For this mark, allow a non-exact value (awrt 1.72)

Stage 2 ALT 2	Use $u = \sqrt{4 - x^2}$ to obtain: $\int_0^{\sqrt{2}} x\sqrt{4-x^2}\,dx = \int_2^{\sqrt{2}} \cancel{x}u\left(-\frac{\sqrt{4-x^2}}{\cancel{x}}du\right)$ $= -\int_2^{\sqrt{2}} u^2 du$ $= \left[-\frac{u^3}{3}\right]_2^{\sqrt{2}}$ $= -\frac{\sqrt{2}^3}{3} + \frac{2^3}{3}$ $= \frac{8}{3} - \frac{2}{3}\sqrt{2}$	M1* A1 M1(dep*) A1 [4 marks max.]	Chooses a suitable substitution, attempts to change the area element and obtains integrand in their new variable Obtains correct integrand (ignore limits) Obtains integral of the form Bu^3 , $B \neq 0$, and substitutes **correct** limits (ft their P) **in the correct order** Obtains correct value for the integral For this mark, allow a non-exact value (awrt 1.72)
Stage 2 ALT 3	Use $x = 2\sin\theta$ to obtain $\int_0^{\sqrt{2}} x\sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{4}} (2\sin\theta)\sqrt{4-(2\sin\theta)^2}(2\cos\theta)d\theta$ $= 8\int_0^{\frac{\pi}{4}} \sin\theta\cos^2\theta$ $= \left[-8\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{4}}$ $= -\frac{8\cos^3\left(\frac{\pi}{4}\right)}{3} + \frac{8\cos^3 0}{3}$ $= \frac{8}{3} - \frac{2}{3}\sqrt{2}$	M1* A1 M1(dep*) A1 [4 marks max.]	**Mark other trig substitutions similarly** Chooses a suitable substitution, attempts to change the area element and obtains integrand in their new variable Obtains correct integrand (ignore limits) Obtains integral of the form $B\cos^3\theta$, $B \neq 0$, and substitutes **correct** limits (ft their P) **in the correct order** Obtains correct value for the integral For this mark, allow a non-exact value (awrt 1.72)