Revision Session 2 – Solutions to Question Set 2

These are mainly hints. View the videos for the detailed worked solutions.

Question 1

(a) Let n be an integer. Write down your three consecutive integers, sum and then conclude.

(b) Let n be an integer. Characterise two consecutive odd numbers, find their difference of their squares and conclude.

Question 2

Using completing the square, we have

$$\begin{aligned} x^2+4x+7 &= (x+2)^2-2^2+7 \\ &= (x+2)^2+3 \end{aligned}$$

Since $(x+2)^2 \geq 0$ for all x, it follows that $(x+2)^2+3>0$ for all x.

$\therefore x^2+4x+7>0$ for all x as required.

Question 3

(a) Substitute 6, 7 and 8 into the expression and show it is positive in each case.

(b) You need to find a value of n for which the expression is not positive.

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Question 4

We need to split this up into cases.

If n is not a multiple of 3, there are only two cases:

Case 1: it is one bigger than a multiple of 3
Case 2: it is two bigger than a multiple of 3

Case 1:

If n is one bigger than a multiple of 3, we can write $n = 3p + 1$ for some integer p.

Then
$$\begin{aligned} n^2 &= (3p+1)^2 \\ &= 9p^2 + 6p + 1 \\ &= 3(3p^2 + 2p) + 1 \end{aligned}$$

which is of the form required. So true in case 1.

Case 2:

If n is two bigger than a multiple of 3, we can write $n = 3p + 2$ for some integer p.

Then

$$\begin{aligned} n^2 &= (3p+2)^2 \\ &= 9p^2 + 12p + 4 \\ &= 9p^2 + 12p + 3 + 1 \\ &= 3(3p^2 + 4p + 1) + 1 \end{aligned}$$

which is of the form required. So true in case 2.

Hence the statement is true, i.e. if n is not a multiple of 3, then n^2 can be written in the form $3k + 1$ for some k.

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Question 5

This is best proven by considering the different cases.

Case 1: n is even.
Case 2: n is odd.

Consider each case separately and show that the expression is even in both cases.
If you get stuck, look at the video solutions.

Question 6

(a) Let p and q be two rational numbers. Then can write $p = \frac{a}{b}$ and $q = \frac{c}{d}$, where $a,b,c,d \in \mathbb{Z}$ and $b, d \neq 0$.

Then $p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, which is rational as required.

(b) For example, consider $\sqrt{2}$ and $1 - \sqrt{2}$.

Question 7

Want to prove that $x + \frac{1}{x} \geq 2$

We know that $(x-1)^2 \geq 0$ for all x. Then:

$\Rightarrow x^2 - 2x + 1 \geq 0$
$\Rightarrow x^2 + 1 \geq 2x$

Since $x > 0$, can divide by x to get $x + \frac{1}{x} \geq 2$ as required.

See the video solutions to get an insight about why we knew to start from $(x-1)^2 \geq 0$ and what a wrong proof would look like.

Question 8

(a) This is the easier part. Write down what it means for n to be even, square and then the result is immediate.

(b) Let n^2 be even. Then can write $n^2 = 2k$ for some integer k.

Consider in more detail:

$$n^2 = 2k$$

This tells us that n^2 is a multiple of 2.
So 2 divides n^2.
If 2 divides n^2, then it must divide one of its factor.
Both of the factors of n^2 are n, so 2 must divide n.

As a result, n must be even.

Question 9

(a) Write down what it means for x to be rational. This is the tricky bit. The rest follows.

(b) No the statement is false. For example, let $y^3 = 2$ which is rational.

But $y = \sqrt[3]{2}$ which is not rational.

Hence the statement is false.

Question 10

(a) We know that $(\sqrt{a}-\sqrt{b})^2 \geq 0 \quad \forall a,b > 0$. Then

$$\Rightarrow a - 2\sqrt{ab} + b \geq 0$$
$$\Rightarrow a + b \geq 2\sqrt{ab}$$
$$\Rightarrow \frac{1}{2}(a+b) \geq \sqrt{ab}$$
$$\Rightarrow \sqrt{ab} \leq \frac{1}{2}(a+b)$$

This is similar to Q7 in terms of knowing where to start. Watch the video for guidance on this sort of thing and what would constitute an incorrect proof.

(b) We just need to substitute those expressions into part (a):

$$\sqrt{2^x \cdot 2^{4-x}} \leq \frac{1}{2}(2^x + 2^{4-x})$$
$$\Rightarrow \sqrt{2^4} \leq \frac{1}{2}(2^x + 2^{4-x})$$
$$\Rightarrow 4 \leq \frac{1}{2}(2^x + 2^{4-x})$$
$$\Rightarrow 2^x + 2^{4-x} \geq 8$$

as required.

(c) Equality holds when $2^x + 2^{4-x} = 8$. This is a 'disguised' quadratic equation. Solve it by factorising it directly, or try to use the substitution $y = 2^x$.

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