Worksheet: **Proof** | CM

1 (a) Prove that $x^2 + 4x + 12 > 0$ for all integers x.

(b) It is claimed that "$n^2 + 3n + 1 > 0$ for all integers n"

Disprove this statement using a suitable counter-example.

2 Emily wants to prove that the sum of the squares of any two odd numbers is even.

Her proof is shown below.

> Let the odd numbers be $2n - 1$ and $2n + 1$ for integers n.
>
> Then
>
> $$\begin{aligned}(2n-1)^2 + (2n+1)^2 &= 4n^2 - 4n + 1 + 4n^2 + 4n + 1 \\ &= 8n^2 + 2 \\ &= 2(4n^2 + 1)\end{aligned}$$
>
> which is a multiple of 2 and so even.
>
> Hence, the sum of the squares of any two odd numbers is even.

Her proof is not correct.

(a) Explain why.

(b) Provide a correct proof of the statement.

3 Let p and q be prime numbers, $p, q \neq k$, where k is an integer.

Given that $p + q$ is always even,

(a) state the value of k.

(b) Prove that $p + q$ is always even for all primes except k.

4 The claim is that if n is an integer, then $q = n^2 - 2$ is not divisible by 4.

(a) Use exhaustion to prove that the statement is true for $4 \leq n \leq 7$.

[The rest of the question is A level only. It is broken down since it is a bit challenging.]

Now we will prove the claim in full.

(b) Consider the case when n is odd. Is n^2 odd or even? What about $n^2 - 2$?
Hence, explain why it cannot be a multiple of 4 in this case.

(c) Use a similar process for the case when n is even to show the claim is not true in this case either.

5 (a) Prove that if n is even, then $(3n)^2$ is even.

(b) Prove that if $(3n)^2$ is even, then n is even.
[Hint for (b): consider factors]

6 **[A level only]**

For all real x, we have that

$$(5x-3)^2+1 \geq (3x-1)^2$$

Prove the statement using a proof by contradiction.

7 **[A level only]**

(a) Prove that the square root of 2 is irrational.

(b) Prove that the square root of 3 is irrational.

(c) Prove that the cube root of 5 is irrational.

8 **[A level only]**

Prove that there are an infinite number of primes.

9 **[A level only]**

(a) Prove that, for all real and positive x, $x+\frac{25}{x} \geq 10$

(b) Give a counter-example to show that the statement is not necessarily true if x is real but negative.

(c) Is the statement ever true if x is negative?

10 Consider the statement

"If $3n^2 + 2n$ is even, then n is even"

In this question, you will prove this in three ways.

Method 1:

(a) Since $3n^2 + 2n$ is even, it can be written in the form $3n^2 + 2n = 2k$ for some integer k.
By considering factors, complete the proof.

Method 2:

(b) Write $3n^2 + 2n = n^2 + (2n^2 + 2n) = 2k$ for some integer k.
Think about what you can deduce about n^2 and complete the proof.

Method 3 [A level only]:

(c) Prove the statement by contradiction.