Proof worksheet – solutions

These are brief, sketched solutions.
Comments in blue can be ignored, but they provide further explanation and outline common misconceptions

Question 1

(a)

$$\begin{aligned} x^2+4x+12 &= (x+2)^2-4+12 \\ &= (x+2)^2+8 \end{aligned}$$

Now $(x+2)^2 \geq 0$ for all x, hence $(x+2)^2+8>0$ for all x.

(b) [Note that the statement will obviously be true if n is positive (since you're just adding positive numbers), so you need to look for your counter-example in the negative numbers.]

Take $n=-1$, then

$$\begin{aligned} (-1)^2+3(-1)+1 &= 1-3+1 \\ &= -1 \\ &< 0 \end{aligned}$$

and so the statement is not true for $n=-1$.

You can also use $n = -2$.

[Remember: n is only defined to be an integer, so providing a non-integer counter-example does not disprove the claim.]

Question 2

(a) her numbers are not independent (they both depend on n), while the question wants you to prove that the sum of the squares of *any* two odd number is even.

(b) Let the two odd numbers be $2n+1$ and $2m+1$ for n, m integers

Then

$$\begin{aligned} (2n+1)^2+(2m+1)^2 &= 4n^2+4n+1+4m^2+4m+1 \\ &= 4n^2+4n+4m^2+4m+2 \\ &= 2(2n^2+2n+2m^2+2m+1) \end{aligned}$$

which is even. Hence, the sum of the squares of any two odd numbers is even.

CM

Question 3

(a) $k = 2$ (all the other primes are then odd and so their sum is always even)

(b) Since the other primes are odd, we just need to prove that the sum of any two odd numbers is even.

Let the odd numbers be $2n + 1$ and $2m + 1$, where n, m are integers.

Then $(2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$, which is even.
Hence, the sum of two odd numbers is always even. In particular, the sum of any two odd primes is even.

Question 4

(a)

n	$q = n^2 - 2$	Divisible by 4?
4	14	no
5	23	no
6	34	no
7	47	no

Hence, the claim is true for $4 \leq n \leq 7$

(b) If n is odd, then n^2 is also odd. So $n^2 - 2$ is also odd.
So $q = n^2 - 2$ cannot be a multiple of 4 since it is odd.

(c) If n is even, then n^2 is even and must be a multiple of 4.
But then $n^2 - 2$ cannot be a multiple of 4 also…

Question 5

(a) Let $n = 2n$, where n is an integer. Then n is even.
$\Rightarrow n^2 = (2n)^2 = 4n^2 = 2(2n^2)$, which is a multiple of 2 and so also even, as required.

(b) Let n^2 be a multiple of 2. Then $3n^2 = 2k$ for some integer k.

So 2 must divide the LHS. Since 2 does not divide 3, it must divide $n^2 \Rightarrow$ it must divide n.

Hence, since 2 divides n, it must be even.

Question 6

Assume for a contradiction that there exists $x \in \mathbb{R}$ such that
$(5x-3)^2+1<(3x-1)^2$

$\Rightarrow 25x^2-30x+9+1<9x^2-6x+1$
$\Rightarrow 16x^2-24x+9<0$
$\Rightarrow (4x-3)^2<0 \quad \#$

which is a contradiction since a square number is never negative.

Hence $(5x-3)^2+1 \geq (3x-1)^2$ for all x as required.

Question 7

(a) Assume for a contradiction that the square root of 2 is rational, i.e. there exists coprime integers $a, b, b \neq 0$, such that $\sqrt{2}=\dfrac{a}{b}$

Then $2=\dfrac{a^2}{b^2} \Rightarrow a^2=2b^2$. So a^2 is a multiple of 2, and so must be a.

Let $a = 2k$ for some integer k, then $(2k)^2=2b^2 \Rightarrow b^2=2k^2$. But this implies that b^2 (and hence b) is a multiple of 2. #

This is a contradiction because a and b were assumed to be coprime (share no prime factors). Hence the square root of 2 must be irrational.

(b) Assume for a contradiction that the square root of 3 is rational, i.e. there exists coprime integers $a, b, b \neq 0$, such that $\sqrt{3}=\dfrac{a}{b}$

Then $3=\dfrac{a^2}{b^2} \Rightarrow a^2=3b^2$. So a^2 is a multiple of 3, and so must be a.

Let $a = 3k$ for some integer k then $(3k)^2=3b^2 \Rightarrow b^2=3k^2$. But this implies that b^2 (and hence b) is a multiple of 3. #

This is a contradiction because a and b were assumed to be coprime (share no prime factors). Hence the square root of 3 must be irrational.

(c) Assume for a contradiction that the cube root of 5 is rational, i.e. there exists coprime integers $a, b, b \neq 0$, such that $\sqrt[3]{5} = \frac{a}{b}$.

Then $5 = \frac{a^3}{b^3} \Rightarrow a^3 = 5b^3$. So a^3 is a multiple of 5, and a must be a multiple of 5 also.

Let $a = 5k$ for some integer k, then $(5k)^3 = 5b^3 \Rightarrow b^3 = 25k^3$. So b^3 (and hence b) must be a multiple of 5. #

This is a contradiction because a and b were assumed to be coprime (share no prime factors). Hence the cube root of 5 must be irrational.

Question 8

Approach 1 (standard):

Assume for a contradiction that the set of primes is finite. Call these primes $p_1, p_2, \ldots, p_k$.

Consider $P = p_1 p_2 ... p_k + 1$. Then P is not divisible by any of $p_1, p_2, \ldots, p_k$, so there must exist another prime (namely P itself) that divides P #

This is a contradiction since we have constructed another prime outside of our supposedly finite set. Thus there must be an infinite number of primes.

Approach 2 (neater?):

Assume for a contradiction that the set of primes is finite. Call these primes $p_1, p_2, \ldots, p_k$. Since the set is finite, the largest prime exists and let p_k be this number.

Consider $P = p_k! + 1$. Then P is not divisible by any number less than or equal to p_k, so P must be prime. #

This is a contradiction since we have constructed another prime outside of our supposedly finite set. Thus there must be an infinite number of primes.

Question 9

(a) Assume for a contradiction that there exists a real and positive x such that
$x+\frac{25}{x}<10$.

Then multiplying by x gives

$$x^2+25<10x \qquad \text{(we have used the positivity of } x \text{ here. How?)}$$

$\Rightarrow x^2-10x+25<0$

$\Rightarrow (x-5)^2<0 \quad \#$

which is a contradiction since a square number is never negative.

Hence $x+\frac{25}{x}\geq 10$ for all positive and real x as required.

Note, despite however tempting it may be or correct it may seem, the following proof is **NOT** valid:

$x+\frac{25}{x}\geq 10$

$\Rightarrow x^2+25\geq 10x$

$\Rightarrow x^2-10x+25\geq 0$

$\Rightarrow (x-5)^2\geq 0$

which is true for all x so the statement must be true.

The issue here is that you have ***assumed*** the statement is true and then gone on to verify a true statement. Statement $\Rightarrow$ true statement does **NOT** necessarily mean the statement is true. The main issue is that the implications are going in the wrong direction. You need to start with a true statement and reach the statement.

That said, the above proof is not entirely useless. In fact it is a good place to start. Check to see if you can reverse the implications to obtain the proof you were looking for (or replace the implications with iff symbols).

(b) Take $x = -1$. Then the LHS $= -1 - 25 = -26 < 10$ as required.

(c) No it is not true for any negative x. This is because the LHS is always negative if x is negative and 10 is bigger than any negative number.

Question 10

(a) $3n^2 + 2n = 2k$ for some integer k since $3n^2 + 2n$ is even

$\Rightarrow n(3n+2) = 2k$
So 2 must divide either n or $3n + 2$.

The first case (2 divides n) proves the statement trivially (this is the definition of n being even)

Consider the second case. Suppose 2 divides $3n + 2$. Then 2 must divide $3n$. But 2 doesn't divide 3, so it must divide n. Again, we have the situation where 2 divides n, i.e. n is even

So in either case, we deduce that n is even. Hence if $3n^2 + 2n$ is even, then n is even.

(b) This is similar (and perhaps a little easier than (a))

Note that 2 must divide $n^2 + 2(n^2 + n)$. 2 already divides the second term, but for it to divide the whole thing, it must also divide n^2. And if 2 divides n^2, it must divide n.

Hence, n is even.

(c) Suppose for a contradiction that n is odd.

Then $n = 2k + 1$ for some integer k.

$$\begin{aligned} 3(2k+1)^2 + 2(2k+1) &= 3(4k^2 + 4k + 1) + 4k + 2 \\ &= 12k^2 + 16k + 5 \\ &[= 2(6k^2 + 8k + 2) + 1] \ \# \end{aligned}$$

which is contradiction since the expression is always odd but $3n^2 + 2n$ is assumed to be even.