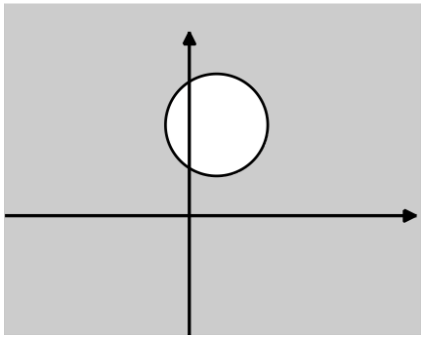
AS Level
Further Maths

Bronze Set A, Paper F2 (Edexcel version)

Question	Solution	Partial Marks	Guidance
1 (a)	0	B1 **[1]**	Cao
1 (b)	e.g. 3	B1 **[1]**	Any number that is one less than a multiple of 4
1 (c)	1001 = 1(663) + 338 663 = 1(338) + 325 338 = 1(325) + 13 325 = 13(25) + 0 (so greatest common divisor d = 13) Then 13 = 338 – 1(325) So 13 = 338 – 1(663 – 1(338) = 2(338) – 1(663) So 13 = 2(1(1001) – 1(663)) – 1(663) = 2(1001) – 3(663) (hence e.g. u = 2, v = –3)	M1* A1 M1(dep*) A1 **[4]**	Uses Euclidean algorithm, reaching the second line shown in the scheme Completes the algorithm correctly. Does not need to state d Starts the extended Euclidean algorithm showing and simplifying at least one back substitutions Completes the algorithm correctly giving values of u and v
2 (a)	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (1-\lambda)^2 - 9 = 0$ $\Rightarrow \lambda^2 - 2\lambda - 8 = 0$	M1 A1 **[2]**	Complete method to find characteristic equation Correct characteristic equation – may use another variable
2 (b)	By Cayley-Hamilton, $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = 0$ $\Rightarrow \mathbf{A}^3 - 2\mathbf{A}^2 - 8\mathbf{A} = 0$ $\Rightarrow \mathbf{A}^3 - 2(2\mathbf{A} + 8\mathbf{I}) - 8\mathbf{A} = 0$ $\Rightarrow \mathbf{A}^3 = 4\mathbf{A} + 16\mathbf{I} + 8\mathbf{A}$ So $\mathbf{A}^3 = 12\mathbf{A} + 16\mathbf{I}$ as required **AG**	B1ft M1 A1 **[3]**	Uses Cayley-Hamilton on their characteristic equation, replacing the variable with **A** and multiplying any constant terms by **I** Multiplies equation by **A** Replaces $\mathbf{A}^2$ correctly to obtain the given answer convincingly and with no errors seen

2 (c)	Eigenvalues are solutions to $\lambda^2 - 2\lambda - 8 = 0$ $(\lambda - 4)(\lambda + 2) = 0 \Rightarrow$ eigenvalues are 4 and –2 To find eigenvectors, need to solve $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$ For $\lambda = 4$:, $\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$, so $x = y$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector For $\lambda = -2$:, $\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$, so $x = -y$. Hence $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector	B1 M1 A1 A1 **[4]**	Correct eigenvalues Complete method to find an eigenvector corresponding to one of their eigenvalues A correct eigenvector corresponding to the eigenvalue 4 A correct eigenvector corresponding to the eigenvalue –2
2 (d)	$\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$ Matrix of eigenvectors $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, but since **P** should be orthogonal it will be the matrix of normalised eigenvectors, i.e. $\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$	B1ft M1 A1 **[3]**	Forms diagonal matrix for **D** with their eigenvalues along the diagonal and 0s elsewhere Forms a matrix ft their eigenvectors (need not be normalised for this mark). If an attempt is made to normalise the eigenvectors, allow the ft if slips are made when normalising Correct and consistent matrices for **D** and **P** with **P** orthogonal
2 (d) **ALT**	$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ (or $\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$) $\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$	B1ft M1 A1 **[3]**	Forms a matrix ft their eigenvectors (need not be normalised for this mark). If an attempt is made to normalise the eigenvectors, allow the ft if slips are made when normalising Forms diagonal matrix for **D** with their eigenvalues along the diagonal and 0s elsewhere **OR** complete ft calculation of $\mathbf{PAP}^{-1}$ Both **D** and **P** correct with **P** orthogonal

3 (a)	$\lvert x+\mathrm{i}y+2\rvert = 4\lvert x+\mathrm{i}y-3\mathrm{i}\rvert$ $\Rightarrow \lvert (x+2)+\mathrm{i}y\rvert = 4\lvert x+\mathrm{i}(y-3)\rvert$ $\Rightarrow \sqrt{(x+2)^2+y^2} = 4\sqrt{x^2+(y-3)^2}$ $\Rightarrow (x+2)^2+y^2 = 16\left[x^2+(y-3)^2\right]$ $\Rightarrow x^2+4x+4+y^2 = 16(x^2+y^2-6y+9)$ $\Rightarrow 15x^2-4x+15y^2-96y+140=0$ $\Rightarrow x^2-\frac{4}{15}x+y^2-\frac{32}{5}y+\frac{28}{3}=0$ $\Rightarrow \left(x-\frac{2}{15}\right)^2-\frac{4}{225}+\left(y-\frac{16}{5}\right)^2-\frac{256}{25}+\frac{28}{3}=0$ $\Rightarrow \left(x-\frac{2}{15}\right)^2+\left(y-\frac{16}{5}\right)^2=\frac{208}{225}$ Therefore the locus of P is a circle with centre $\left(\frac{2}{15},\frac{16}{5}\right)$ and radius $\frac{4\sqrt{13}}{15}$	M1[1] M1[2](dep 1) M1[3](dep 2) A1 M1(dep 3) A1 **[6]**	Replaces z with x + iy and combines real and imaginary parts Uses definition of the modulus Squares both sides and attempts to expand out and collect terms Correct expanded and simplified expression For completing the square on their equation of the circle Complete and convincing proof demonstrating that the locus is a circle and giving the correct centre and radius of the circle. Accept equivalent forms for the coordinates and the radius including decimals which are given correct to 3sf
3 (b)		B1ft B1 **[2]**	Circle drawn on Argand diagram in correct position ft their centre and radius (no markings needed) For the ft, the candidate's centre and radius must be consistent, i.e. check where the circle crosses the axes Region outside of their circle shaded

4 (a)	(Let $A \in S$, then) (det $A =$) $\begin{vmatrix} x & x \\ x & x \end{vmatrix} = x^2 - x^2 = 0$, therefore any matrix in S is singular	B1 **[1]**	Shows that the determinant of a matrix with identical entries is 0 + conclusion, i.e. ' therefore any matrix in S is singular', 'as required', 'qed' etc.
4 (b)	$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \Rightarrow \begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ $\Rightarrow 2xe = x$, so $e = \frac{1}{2}$	M1 A1 **[2]**	Multiplies matrices together and equates entries Correct value of e
4 (c)	$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\Rightarrow 2xy = \frac{1}{2}$, so $y = \frac{1}{4x}$	M1 A1 **[2]**	Multiplies matrices together and equates entries using their value of e Correct value of y in terms of x
4 (d)	Part (b) shows that there is an identity element (in S) Part (c) shows that each matrix (in S) has an inverse (in S) Let $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \in S$, $a \neq 0$ and $B = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \in S$, $b \neq 0$, $a \neq b$, then $AB = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \in S$, hence S is closed under (matrix) multiplication Matrix multiplication is associative (by assumption), so S forms a group (under matrix multiplication)	B1 B1 M1 A1 **[4]**	States that part (b) shows that there is an identity element (in S) States that part (c) shows that each matrix has an inverse (in S) Multiplies two <u>distinct</u> matrices with identical elements **and** shows that the product of these matrices also has identical elements Need not say $a \neq b$: if they use different letters, this is implied Need not specify that A, B are in S or $a, b \neq 0$ Complete and convincing proof with each condition verified For this final mark, they must have made a comment about the product AB having identical elements or stated that it is also in S **and** then concluded that S is <u>closed</u> under (matrix) multiplication

4 (e)	If $x = 0$, then some matrices will not have a (well-defined) inverse	B1 **[1]**	Identifies that $x \neq 0$ is necessary so that each element has an inverse (that is well-defined)
5 (a)	When $n = 1$, $u_1 = 2^1(2^2 - 3) = 2(1) = 2$, so true for $n = 1$ Assume true when $n = k$, i.e. $u_k = 2^k\left(2^{k+1} - 3\right)$ Then $u_{k+1} = 2u_k + 4^{k+1}$ $= 2\left[2^k(2^{k+1} - 3)\right] + 4^{k+1}$ $= 2^{k+1}(2^{k+1} - 3) + 2^{2k+2}$ $= 2^{k+1}\left(2^{k+1} - 3 + 2^{k+1}\right)$ $= 2^{k+1}\left(2(2^{k+1}) - 3\right)$ $= 2^{k+1}\left(2^{k+2} - 3\right)$ Therefore, if true for $n = k$, it has been shown to be true for $n = k + 1$. Since true for $n = 1$, it follows by induction that it is true for all $n \in \mathbb{Z}^+$	B1 M1* M1(dep*) A1 A1 **[5]**	Shows the statement is true for $n = 1$ Makes the assumption (can be implied) Uses their assumption and makes some attempt at getting to the required result Shows statement is true for $n = k + 1$. All the above steps do not need to be shown but you should be convinced – look out for fudged workings Completely correct proof with no errors seen and a conclusion containing all the underlined elements