AS Level
Further Maths

Bronze Set A, Core Pure (Edexcel version)

Question	Solution	Partial Marks	Guidance
1 (a)	$\alpha+\beta=\frac{5}{2}$ and $\alpha\beta=3$	B1 B1 **[2]**	Correct values (1 mark for each correct value)
1 (b)	$(\alpha+2)(\beta+2)=\alpha\beta+2(\alpha+\beta)+4$ $=3+2\left(\frac{5}{2}\right)+4$ $=12$	M1 A1ft **[2]**	Expands brackets and substitutes in their values from (a) into a correct expression correctly Correct value ft their (a)
1 (c)	Sum of roots = $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{5}{2}\right)^2-2(3)=\frac{1}{4}$ Product of roots is $\alpha^2\beta^2=(\alpha\beta)^2=9$ So equation is $x^2-\frac{1}{4}x+9=0$ $\Rightarrow 4x^2-x+36=0$ so $a=-1$, $b=36$	M1 M1 A1 A1 **[4]**	Complete method to find the value of the sum of the roots using their values from (a) Complete method to find the product of the roots using their values from (a) Correct value for the product of the roots and the sum of the roots Correct values of a and b stated Give A1 BOD if correct equation obtained and values not stated. Do **not** ISW if correct equation stated and they go on to state wrong values for a and b
2 (a)	Modulus of z is 5 Argument of z is $-\pi+\tan^{-1}\left(\frac{3}{4}\right)=-2.4980...$ So $z=5(\cos(-2.50)+\mathrm{i}\sin(-2.50))$	B1 M1 A1 **[3]**	Correct modulus Method to find the argument of z including consideration of quadrant (allow incorrect argument for tan, e.g. 4/3 instead of 3/4) Correct mod-arg form of z with arg given to 3sf in radians Allow 5 to be distributed

2 (b)	$z_1z_2 = (-4-3\text{i})(a+7\text{i})$ $= -4a - 28\text{i} - 3a\text{i} + 21$ $= (21-4a) + \text{i}(-28-3a)$	M1 A1 **[2]**	Multiplies z_1z_2 to obtain the correct number of terms **and** uses $\text{i}^2 = -1$ Correct answer in the correct form **Allow** $(21-4a) - \text{i}(28+3a)$
2 (c)	If z_1z_2 is real, then $-28-3a=0 \Rightarrow a = -\dfrac{28}{3}$	M1 A1ft **[2]**	Sets imaginary part of their (b) equal to 0 Correct value of a ft their (b)
2 (d)	$21-4a=0 \Rightarrow a = \dfrac{21}{4}$	M1 A1 **[2]**	Sets real part of (b) equal to 0 Correct value of a ft their (b)
3 (a)	$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}+\mathbf{k}) = 3-4+1=0$ $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 1+1-2=0$ hence (since the dot product is 0), **i** + **j** + **k** is perpendicular to 3**i** – 4**j** + **k** and **i** + **j** – 2**k**	M1 M1 A1 **[3]**	Attempts to find dot product between **i** + **j** + **k** and 3**i** – 4**j** + **k** Attempts to find dot product between **i** + **j** + **k** and **i** + **j** – 2**k** Conclusion Allow a conclusion after each dot product is shown to equal 0 in place of one at the end **ALT**: Use of cross product to show perp. vector is in the direction of **i** + **j** + **k** M1 – complete method to find cross product between 3**i** – 4**j** + **k** and **i** + **j** – 2**k** (in any order) A1 – two components correct A1 – all three components correct + conclusion
3 (b)	Plane contains the point (1, –2, 3) $1(1)+1(-2)+1(3)=2$ So equation of the plane is $x+y+z=2$	B1 M1 A1 **[3]**	Seen or implied Attempts to find d using their point Correct equation of the plane

4	$w = x + 1$, so $x = w - 1$ Substituting gives $(w-1)^3 - 9(w-1)^2 + 2(w-1) + 1 = 0$ $w^3 - 3w^2 + 3w - 1 - 9w^2 + 18w - 9 + 2w - 2 + 1 = 0$ $\Rightarrow w^3 - 12w^2 + 23w - 11 = 0$	B1 M1* M1(dep*) A1 A1 **[5]**	Makes a connection between x and w writing $x = w - 1$ Substitutes their $w - 1$ into the given cubic Expands brackets attempting to obtain answer in the required form Any two of p, q and r correct Completely correct equation
5	$600\pi = \pi\int_1^k x^3\, dx$ $\Rightarrow 600 = \left[\frac{1}{4}x^4\right]_1^k$ $\Rightarrow 600 = \frac{1}{4}k^4 - \frac{1}{4}$ $\Rightarrow k^4 = 2401$ $\Rightarrow k = 7$	B1 M1* A1 M1(dep*) A1 **[5]**	Forms the correct **equation** in any form at any point (can be implied) Integrates indefinitely their y^2 Correct indefinite integration of the correct y^2 (no ft) Substitutes in the limits of 1 and k in the correct order **and** attempts to solve for k Correct value of k. A0 if they write $k = \pm 7$
6 (a)	$\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$	B1 B1 B1 **[3]**	Correct matrices (one mark for each correct matrix)
6 (b)	$\mathbf{A}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	B1 **[1]**	Correct matrix

6 (c)	$\mathbf{B}^1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, so true for $n = 1$ Assume true for $n = k$, i.e. $\mathbf{B}^k = \begin{pmatrix} 1 & 1 & 2k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Then for $n = k + 1$, $\mathbf{B}^{k+1} = \mathbf{B}^k\mathbf{B}^1$ $= \begin{pmatrix} 1 & 1 & 2k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 1+1+2k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ * $= \begin{pmatrix} 1 & 1 & 2k+1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 2(k+1)-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ so true for $n = k + 1$ If true for $n = k$, then by induction it has been shown to be true for $n = k + 1$. Since true for $n = 1$, it is true for all positive integers n.	B1 M1* M1(dep*) A1 A1 **[5]**	Shows the statement is true for n =1 Makes the assumption and shows intention to multiply $\mathbf{B}^k$ by $\mathbf{B}$ (either way around) Multiplies the matrices obtaining at least seven correct entries Convincingly shows the statement is true for $n = k + 1$ (the second line can be omitted) They must obtain $2(k + 1) - 1$ or make it clear that $2k + 1$ is the form required Complete proof with no errors seen and conclusion with all the underlined elements (or equivalent)

7 (a)	For $n = 1$, LHS = $\sum_{r=1}^{1} r^2 = 1^2 = 1$ RHS = $\frac{1}{6}(1+1)(2+1) = 1$ So LHS = RHS and statement is true for $n = 1$	B1	Shows the statement is true for $n = 1$
	Assume true for $n = k$, i.e. $\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$	M1*	Makes the assumption (can be implied) and shows intention to use it in a correctly partitioned sum
	Then $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$ $= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$ $= \frac{(k+1)}{6}(2k^2 + k + 6k + 6)$ $= \frac{(k+1)}{6}(2k^2 + 7k + 6)$ $= \frac{(k+1)}{6}(k+2)(2k+3)$ $= \frac{(k+1)}{6}((k+1)+1)(2(k+1)+1)$	M1(dep*)	Uses the assumption and extracts a factor of $(k + 1)$
	so true for $n = k + 1$	A1	Convincingly shows the statement is true for $n = k + 1$ If they stop at the penultimate line, they must make it clear that this is the form required
	<u>If true for $n = k$</u>, then by induction it has been shown to be <u>true for $n = k + 1$</u>. <u>Since true for $n = 1$</u>, it is <u>true for all positive integers n.</u>	A1 **[5]**	Complete proof with no errors seen and conclusion with all the underlined elements (or equivalent)

7 (b)	$4\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1 = \frac{4}{6}n(n+1)(2n+1) - n$ $= \frac{1}{3}n(2(n+1)(2n+1) - 3)$ $= \frac{1}{3}n(4n^2 + 6n + 2 - 3)$ $= \frac{1}{3}n(4n^2 + 6n - 1)$ **AG**	B1 M1 A1 **[3]**	Sum of 1 from $r = 1$ to $n = n$ seen or implied Uses linearity of the sum, the result from (a) to find an expression for the sum and takes out a factor of $n/3$ Shows the result convincingly Use of induction is 0/3 as they are asked to use part (a)
7 (c)	$\sum_{r=10}^{30}(4r^2 - 1) = \sum_{r=1}^{30}(4r^2 - 1) - \sum_{r=1}^{9}(4r^2 - 1)$ $= \frac{1}{3}(30)(4(30)^2 + 6(30) - 1) - \frac{1}{3}(9)(4(9)^2 + 6(9) - 1)$ * $= 37790 - 1131$ * $= 36659$	M1 A1 **[2]**	Partitions the sum (can be implied) Obtains the result convincingly (**one** of the lines marked with * can be omitted) Answer only is 0/2 (this includes the direct use of a calculator without any working shown)
8 (a)	$\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$	B1 **[1]**	Correct matrix
8 (b)	T is represented by $\mathbf{BA} = \begin{pmatrix} 1 & 3 \\ 5 & k \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 25 & k \end{pmatrix}$ Since T_1 and T_2 have the same area, det(**BA**) = ± 1 So either $5k - 75 = 1$ or $5k - 75 = -1$ $\Rightarrow k = \frac{76}{5}$ or $k = \frac{74}{5}$	M1 B1 M1 A1 A1 **[5]**	Attempts to find the matrix **BA** Recognises their matrix **BA** (or **AB**) has determinant +1 or –1 Finds the determinant of their **BA** (or **AB**) and sets it equal to + 1 or –1 One correct value of k Second value of k correct

9 (a)	$f(13) = 13^3 - 23(13)^2 + 299(13) - 2197$ $= 2197 - 3887 + 3887 - 2197$ $= 0$ therefore since f(13) = 0, $(z - 13)$ is a factor of $f(z)$	M1 A1 **[2]**	Substitutes 13 into f showing substitution explicitly Obtains 0 and conclusion, e.g. 'therefore, $(z - 13)$ is a factor', 'as required', 'qed' etc
9 (b)	$f(z) = (z - 13)(z^2 + az + b)$ $= (z - 13)(z^2 - 10z + 169)$ $z^2 - 10z + 169 = 0$ $\Rightarrow (z - 5)^2 - 25 + 169 = 0$ $\Rightarrow (z - 5)^2 = \pm\sqrt{-144}$ $\Rightarrow z = 5 \pm 12i$ So roots of the equation are 13, 5 + 12i and 5 – 12i $\lvert 13 \rvert = 13$ $\lvert 5 + 12i \rvert = \sqrt{5^2 + 12^2} = 13$ $\lvert 5 - 12i \rvert = \sqrt{5^2 + (-12)^2} = 13$ so roots lie on the circle $\lvert z \rvert = 13$	M1[1] A1 M1[2](dep1) M1[3](dep2) A1 M1(dep3) A1 **[7]**	Attempts to find the other factor of $f(z)$ (can use long division) Correct other factor of $f(z)$ Method to find the roots of their other factor of $f(z)$ The other factor must have complex roots Uses $\sqrt{-1} = i$ Correct complex roots of $f(z) = 0$ Method to find the magnitude of the roots Shows all the roots have magnitude 13 and states they lie on the circle $\lvert z \rvert = 13$ or states that $k = 13$
10 (a)	$\cos\theta = \dfrac{(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})}{\sqrt{1^2 + 2^2 + 1^2}\sqrt{2^2 + 3^2 + (-2)^2}}$ $\cos\theta = \dfrac{6}{\sqrt{6}\sqrt{17}}$ $\Rightarrow \theta = 53.552...$	B1 M1 A1 **[3]**	State or imply a correct normal to one of the planes, g.g. $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ or $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ Complete method to find the angle between the planes Correct angle Allow radian equivalent (0.9346…)

10 (b)	Equations of planes are $x+2y+z=1,\ \ 2x+3y-2z=-2$ At z = 2, we have $x+2y=-1,\ \ 2x+3y=2$ Solving the equations simultaneously gives $x=7, y=-4$ So line passes through (7, –4, 2) Hence eq. of line is e.g. $\mathbf{r}=\begin{pmatrix}7\\-4\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\3\end{pmatrix}$	M1* A1 A1ft **[3]**	Substitutes z = 2 into their equations for the two planes Correct values of x and y Correct equation of the line ft their point (or equivalent)
10 (c) (i)	General point on the line is $\begin{pmatrix}7+5\lambda\\-4-\lambda\\2+3\lambda\end{pmatrix}$ Vector between any point on the line and (4, 2, –1) is $\begin{pmatrix}7+5\lambda\\-4-\lambda\\2+3\lambda\end{pmatrix}-\begin{pmatrix}4\\2\\-1\end{pmatrix}=\begin{pmatrix}3+5\lambda\\-6-\lambda\\3+3\lambda\end{pmatrix}$ Point on line closest to (4, 2, –1) satisfies $\begin{pmatrix}5\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3+5\lambda\\-6-\lambda\\3+3\lambda\end{pmatrix}=0\Rightarrow 15+25\lambda+6+\lambda+9+9\lambda=0$ $\Rightarrow\lambda=-\frac{6}{7}$ So the point is $\left(\frac{19}{7},-\frac{22}{7},-\frac{4}{7}\right)$	M1* M1**(dep*) A1 M1(dep**) A1 **[5]**	Finds the direction of the vector between any point on the line and (4, 2, –1). Can be implied (ft their (b)) Forms an equation to find point on the line closest to (4, 2, –1) and attempts to solve it for the parameter (ft their (b)) Correct value of λ (or their parameter) Method to use their parameter to find the coordinates of the point on the line (ft their (b)) Correct point on the line

10 (c) (i) ALT	General point on the line is $\begin{pmatrix}7+5\lambda\\-4-\lambda\\2+3\lambda\end{pmatrix}$ Vector between any point on the line and $(4, 2, -1)$ is $\begin{pmatrix}7+5\lambda\\-4-\lambda\\2+3\lambda\end{pmatrix}-\begin{pmatrix}4\\2\\-1\end{pmatrix}=\begin{pmatrix}3+5\lambda\\-6-\lambda\\3+3\lambda\end{pmatrix}$ So distance *d* satisfies $d^2=(3+5\lambda)^2+(-6-\lambda)^2+(3+3\lambda)^2$ $\Rightarrow d^2=9+30\lambda+25\lambda^2+36+12\lambda+\lambda^2+9+18\lambda+9\lambda^2$ $\Rightarrow d^2=35\lambda^2+60\lambda+54$ Hence $\frac{\mathrm{d}}{\mathrm{d}\lambda}(d^2)=70\lambda+60$ At minimum, $70\lambda+60=0\Rightarrow\lambda=-\frac{6}{7}$ So the point is $\left(\frac{19}{7},-\frac{22}{7},-\frac{4}{7}\right)$	M1* M1**(dep*) A1 M1(dep**) A1 **[5]**	Finds the direction of the vector between any point on the line and (4, 2, –1). Can be implied (ft their (b)) Finds a general expression for the distance from the line to the point and uses a complete method to find the value of the parameter for which this is minimum (ft their (b)) Correct value of λ (or their parameter) Method to use their parameter to find the coordinates of the point on the line (ft their (b)) Correct point on the line
10 (c) (ii)	Shortest distance is thus $\sqrt{\left(4-\frac{19}{7}\right)^2+\left(2--\frac{22}{7}\right)^2+\left(-1--\frac{4}{7}\right)^2}$ $=5.318...$	M1 A1 **[2]**	Method to find the shortest distance between (4, 2, –1) and their (c/i) Correct shortest distance