A Level
Further Maths

Bronze Set A, Paper CP2 (Edexcel version)

Question	Solution	Partial Marks	Guidance
1	$\sin y = 4x \Rightarrow \cos y \dfrac{dy}{dx} = 4$	M1*	Correct implicit differentiation. Allow ±cos*y* on LHS
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{4}{\cos y}$		
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{4}{\sqrt{1-\sin^2(\arcsin(4x))}}$	M1(dep*)	Rearranges for *dy*/*dx* and replaces *y*
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{4}{\sqrt{1-16x^2}}$ **AG**	A1 **[3]**	Complete and convincing proof with no errors seen
2 (a)	$\alpha\beta = -\dfrac{1}{4}$, $\alpha + \beta = -\dfrac{9}{4}$	B1 B1	One mark for each correct value, seen or implied
	$\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$		
	$= \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$		
	$= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$	M1*	Correct expression
	$= \dfrac{\left(-\dfrac{9}{4}\right)^2 - 2\left(-\dfrac{1}{4}\right)}{\left(-\dfrac{1}{4}\right)^2}$	M1(dep*)	Substitutes their values into the correct expression
	$= 89$	A1 **[5]**	Correct final answer

2 (b)	$w = 4x - 1 \Rightarrow x = \frac{w+1}{4}$ $4\left(\frac{w+1}{4}\right)^2 + 9\left(\frac{w+1}{4}\right) - 1 = 0$ $\Rightarrow \frac{w^2+2w+1}{4} + \frac{9w+9}{4} - 1 = 0$ $\Rightarrow w^2 + 11w + 6 = 0$	M1* M1(dep*) A1 **[3]**	Substitutes $x = \frac{w+1}{4}$ Expands brackets and attempts to get answer in required form Correct answer in correct form ISW once correct form reached
3 (a)	$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	B1 **[1]**	Correct definition oe
3 (b)	$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$ $= \frac{(e^x+e^{-x})^2}{(e^x+e^{-x})^2} - \frac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ $= 1 - \frac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ $= 1 - \tanh^2(x)$ **AG**	M1 A1 A1 **[3]**	Attempts to use the quotient or product rule Correct unsimplified derivative Convincingly shows the derivative is $1 - \tanh^2(x)$ Allow if they show it is $\text{sech}^2(x)$, but they must state/have stated that $\text{sech}^2(x)$ is indeed the required $1 - \tanh^2(x)$
3 (c)	$f(0) = 3$ $f'(x) = 1 - \tanh^2 x - \frac{3}{x+1} + 4$, so $f'(0) = 2$ $f''(x) = -2\tanh x(1 - \tanh^2 x) + \frac{3}{(x+1)^2}$, so $f''(0) = 3$ Hence $f(x) = 3 + 2x + \frac{3}{2}x^2 + \ldots$	B1 M1 M1 A1 A1ft **[5]**	Correct value of f(0) Attempts to find the first derivative and evaluate it at 0 Attempts to find the second derivative and evaluate it at 0 Correct first and second derivatives at 0 Correct Maclaurin expansion of f ft their 3 and the values of their derivatives <u>at 0</u>

4 (a) (i)	Rotations preserve volume (so ± 1) Rotations (also) preserve orientation (so + 1)	B1 B1 **[2]**	Volume kept the same under rotation **condone "area"** Orientation kept the same under rotation
4 (a) (ii)	$\det \mathbf{A} = \cos\theta \begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ -\sin\theta & \cos\theta \end{vmatrix} + \sin\theta \begin{vmatrix} 0 & 1 \\ -\sin\theta & 0 \end{vmatrix}$ $= \cos^2\theta + \sin^2\theta$ $= 1$	M1 M1 A1 **[3]**	Correct expression for det**A** Correct method to find the determinant of one of their sub-2x2 matrices Obtains final answer 1 convincingly
4 (b)	Reflection in y axis given by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ Hence T given by $\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & -\sqrt{3} \end{pmatrix}$ $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{3} \\ 1 \\ 1+2\sqrt{3} \end{pmatrix}$ So coordinates of Q are $(2-\sqrt{3},\ 1,\ 1+2\sqrt{3})$	B1 B1ft M1 A1 **[4]**	Correct reflection matrix seen or implied Correct <u>expression</u> for the matrix **M** ft their reflection matrix Interested in the intention to multiply the **correct** rotation and their reflection matrix in the correct order Complete method to find the coordinates of Q using their **M** Correct coordinates Must be in coordinate form or stated as $x = \ldots, y = \ldots, z = \ldots$

4 (c)	$\overrightarrow{PQ}=\begin{pmatrix}2-\sqrt{3}\\1\\1+2\sqrt{3}\end{pmatrix}-\begin{pmatrix}2\\1\\-4\end{pmatrix}=\begin{pmatrix}-\sqrt{3}\\0\\5+2\sqrt{3}\end{pmatrix}$	M1	Method to find direction vector of l_1 using their coordinates of Q
	So the line l_1 has equation e.g. $\mathbf{r}=\begin{pmatrix}2\\1\\-4\end{pmatrix}+\lambda\begin{pmatrix}-\sqrt{3}\\0\\5+2\sqrt{3}\end{pmatrix}$	A1	A correct equation for the line l_1 seen or implied
	(Direction vectors are different, so) the lines are not parallel	B1 ft	States that the lines are not parallel at any stage Allow a ft here if their l_1 is not parallel to l_2
	$2-\lambda\sqrt{3}=2+\mu$ $1=3+2\mu$ $-4+\lambda(5+2\sqrt{3})=-1-3\mu$ Second equation implies $\mu=-1$	M1	Equates components of their l_1 and l_2 and method to solve two of the equations for one of the variables
	Putting this in the first equation gives $\Rightarrow 2-\lambda\sqrt{3}=1$ $\Rightarrow \lambda=\dfrac{1}{\sqrt{3}}$	A1	Obtains correct value of μ and a correct value of λ
	But putting this in the third equation gives $\Rightarrow -4+\lambda(5+2\sqrt{3})=2$ $\Rightarrow \lambda=\dfrac{6}{5+2\sqrt{3}}\neq\dfrac{1}{\sqrt{3}}$ so the lines don't intersect and are skew	A1 **[6]**	Verifies that all three lines are not satisfied and the lines fail to intersect
5 (a)	$p+\mathrm{i}q+2(5+2\mathrm{i})=4(p+\mathrm{i}q)-(-1-13\mathrm{i})$ $p+\mathrm{i}q+10+4\mathrm{i}=4p+4\mathrm{i}q+1+13\mathrm{i}$	M1*	Substitutes for z, z_1 and z_2 and cross multiplies
	Equating real components $\Rightarrow p+10=4p+1$, so $p=3$ Equating imaginary components $\Rightarrow q+4=4q+13$, so $q=-3$	M1(dep*) A1 A1 **[4]**	Equates real **and** imaginary components Correct value of p shown convincingly Correct value of q shown convincingly

5 (a) ALT	$\dfrac{p+\mathrm{i}q+10+4\mathrm{i}}{4p+4q\mathrm{i}+1+13\mathrm{i}}=1$ $\Rightarrow \dfrac{(p+10)+\mathrm{i}(q+4)}{(4p+1)+\mathrm{i}(4q+13)}=1$ $\Rightarrow \dfrac{(p+10)+\mathrm{i}(q+4)}{(4p+1)+\mathrm{i}(4q+13)}\times\dfrac{(4p+1)-\mathrm{i}(4q+13)}{(4p+1)-\mathrm{i}(4q+13)}=1$ $\dfrac{(p+10)(4p+1)+(4q+13)(q+4)}{(4p+1)^2+(4q+13)^2}+\ldots$ $\ldots+\mathrm{i}\dfrac{(4p+1)(q+4)-(p+10)(4q+13)}{(4p+1)^2+(4q+13)^2}=1$ Equating real components gives $\dfrac{(p+10)(4p+1)+(4q+13)(q+4)}{(4p+1)^2+(4q+13)^2}=1$ $\Rightarrow -12p^2+33p-12q^2-75q-108=0$ Equating imaginary components gives $\dfrac{(4p+1)(q+4)-(p+10)(4q+13)}{(4p+1)^2+(4q+13)^2}=0$ $\Rightarrow 3p-39q-126=0$ Solving the two equations, we find p = 3, q = –3 (the other values are not valid)	M1* M1(dep*) A1 A1 **[4]**	Substitutes for z, z_1 and z_2 and attempts to realise the denominator. Condone sign errors Realises the denominator and equates their real and imaginary components Obtains p = 3 from correct equations Obtains q = –3 from correct equations Candidates need **not** include any more steps than shown in the scheme in this case
5 (a) ALT	$\dfrac{z+2z_1}{4z-z_2}=\dfrac{3-3\mathrm{i}+10+4\mathrm{i}}{12-12\mathrm{i}+1+13\mathrm{i}}$ $=\dfrac{13+\mathrm{i}}{13+\mathrm{i}}=1$ as required	M1 M1A1 A1 **[4]**	Substitutes for z, z_1 and z_2 with p and q substituted Shows LHS = 1 (this is either M1A1 or M0A0) Conclusion, e.g. “as required”, “hence, p = 3, q = –3”, “qed”, etc.

5 (b)	$\|z\| = \sqrt{3^2+(-3)^2} = 3\sqrt{2}$ $\arg z = -\tan^{-1}\left(\frac{3}{3}\right) = -\frac{\pi}{4}$ So $z = 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$	B1 M1 A1 oe **[3]**	Correct modulus of z Complete method to find the argument of z Correct expression for z Allow equivalent values for the argument
5 (c)	$r^4 = 3\sqrt{2} \Rightarrow r = (3\sqrt{2})^{\frac{1}{4}}$ $4\theta = -\frac{9\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}$ So $\theta = -\frac{9\pi}{16}, -\frac{\pi}{16}, \frac{7\pi}{16}, \frac{15\pi}{16}$ Hence roots are $w = (3\sqrt{2})^{\frac{1}{4}}\left(\cos\left(-\frac{9\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{16}\right)\right)$, $w = (3\sqrt{2})^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{\pi}{16}\right)\right)$, $w = (3\sqrt{2})^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)$, $w = (3\sqrt{2})^{\frac{1}{4}}\left(\cos\left(\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)$	M1 A1 A1 A1 **[4]**	Clear attempt at both r and θ with at least two different values for arg$w$ Three values of θ correct All four values of θ correct Range not specified, so in particular, allow $\theta = \frac{7\pi}{16}, \frac{15\pi}{16}, \frac{23\pi}{16}, \frac{31\pi}{16}$ Correct complex numbers w ISW once a correct form is reached Allow answers given in polar form Allow modulus of w given to 3sf instead
5 (d)	$a = (3\sqrt{2})^{\frac{1}{4}}$	B1 ft **[1]**	Correct value of a ft their modulus of w

6 (a)	Let $x = 3\sin\theta$, then $dx = 3\cos\theta \, d\theta$ and $\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}(3\cos\theta) d\theta$ $= \int \frac{3\cos\theta}{3\cos\theta} d\theta$ $= \int d\theta$ $= \theta + C$ $= \sin^{-1}\left(\frac{x}{3}\right) + C$ **AG**	M1* M1(dep*) A1 **[3]**	Substitutes in $x = 3\sin\theta$ and uses $dx = \pm 3\cos\theta \, d\theta$ $3\sin^2\theta$ in denominator is M1 BOD Uses $\sin^2\theta + \cos^2\theta = 1$ Shows the result convincingly Condone other letters/expressions for the constant, e.g. k, c, 'const.'
6 (a) ALT	Let $x = 3\cos\theta$, then $dx = 3\sin\theta \, d\theta$ and $\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{9-9\cos^2\theta}}(-3\sin\theta) \, d\theta$ $= \int -\frac{3\sin\theta}{3\sin\theta} d\theta$ $= -\int d\theta$ $= -\theta + c$ $= -\cos^{-1}\left(\frac{x}{3}\right) + c$ $= \sin^{-1}\left(\frac{x}{3}\right) + C$ **AG**	M1* M1(dep*) A1 **[3]**	Substitutes in $x = 3\cos\theta$ and uses $dx = \pm 3\sin\theta \, d\theta$ $3\cos^2\theta$ in denominator is M1 BOD Uses $\sin^2\theta + \cos^2\theta = 1$ Shows the result convincingly Need to either see justification of the fact that $\sin^{-1}x$ and $-\cos^{-1}x$ differ by a constant (a proof is not required, a statement is fine). This can be implied if they change their constant, i.e. from c to C Condone other letters/expressions for the constant, e.g. k, c, 'const.'

6 (b)	Integrating factor is $e^{\int \frac{1}{t}dt = e^{\ln t}} = t$ So $xt = \int t \times \frac{1}{t\sqrt{9-t^2}}\,dt = \sin^{-1}\left(\frac{t}{3}\right) + C$ Using initial conditions, we have $0 = \sin^{-1}\left(\frac{1}{6}\right) + C$, so $C = -\sin^{-1}\left(\frac{1}{6}\right)$ Hence $xt = \sin^{-1}\left(\frac{t}{3}\right) + \sin^{-1}\left(\frac{1}{6}\right)$	B1 M1* A1 M1(dep*) A1 **[5]**	Correct integrating factor Allow $e^{\ln t}$ for this mark Uses their integrating factor to write xt= integral in terms of t Correct general solution in any form without integrals Correct general solution by inspection scores B1 M1 A1 Complete method to use the initial conditions to find their constant Correct solution ISW once correct answer seen in any form Allow 0.167 for the constant but use of degrees is M0
7 (a)	$V = \pi\int_1^\infty y^2\,dx$ $= \pi\int_1^\infty \left(\sqrt{\frac{x+1}{x^3}}\right)^2 dx$ $= \pi\int_1^\infty \left(\frac{x+1}{x^3}\right) dx$ $= \pi\int_1^\infty \left(\frac{1}{x^2} + \frac{1}{x^3}\right) dx$ **AG**	M1 A1 **[2]**	Correct expression for V using formula for volume of revolution Convincingly obtains the required result showing at least one intermediate step
7 (b)	The upper limit of the integral is ∞	B1 **[1]**	Comments on the fact that the upper limit/one of the limits of the integral is infinity
7 (c)	$V = \pi\lim_{b\to\infty}\left[-\frac{1}{x} - \frac{1}{2x^2}\right]_1^b$ $= \pi\lim_{b\to\infty}\left[-\frac{1}{b} - \frac{1}{2b^2} + 1 + \frac{1}{2}\right]$ $= \pi\left[0 + 0 + \frac{3}{2}\right]$ $= \frac{3\pi}{2}$ (So V is finite and $V = 3\pi/2$ as required)	B1 M1 A1 **ft** A1 **[4]**	Correct indefinite integral of the curve Substitutes limits into their integral If a dummy variable is used, an intention to apply a correct limiting process should be seen/implied Correct limiting value of **their** indefinite integral seen Correct value of V (final statement shown in scheme not required)

8 (a)	$y = r\sin\theta = a\sqrt{\sin(2\theta)}\sin\theta$ so $\frac{dy}{d\theta} = \frac{a\sin\theta(2\cos 2\theta)}{2\sqrt{\sin 2\theta}} + a\cos\theta\sqrt{\sin 2\theta}$ At P, $\frac{dy}{d\theta} = 0 \Rightarrow \frac{\sin\theta\cos 2\theta}{\sqrt{\sin 2\theta}} = -\cos\theta\sqrt{\sin 2\theta}$ $\Rightarrow \sin\theta\cos 2\theta = -\cos\theta\sin 2\theta$ $\Rightarrow \sin\theta(\cos^2\theta - \sin^2\theta) = -2\sin\theta\cos^2\theta$ $\Rightarrow \sin\theta(\cos^2\theta - \sin^2\theta + 2\cos^2\theta) = 0$ $\Rightarrow \sin\theta(3\cos^2\theta - \sin^2\theta) = 0$ Since $\sin\theta \neq 0$ at P, must have $3\cos^2\theta - \sin^2\theta = 0$ $\Rightarrow \tan^2\theta = 3 \Rightarrow \theta = \frac{\pi}{3}$ as required **AG**	M1* M1(dep *) A1 M1 M1 A1 **[6]**	Obtains the correct y coordinate Attempts to find $dy/d\theta$. Should see use of product rule and use of chain rule on √sin(2θ) term. Sin/cos to become ±cos/±sin. The 2 from the differentiation of 2θ may be omitted Correct differentiation Sets their $dy/d\theta = 0$ ***OR* SC1:** substitutes $\pi/3$ in to show $dy/d\theta = 0$ Uses double angle formulae for sin and cos to obtain an expression in sinθ and cosθ ***OR*** uses $\sin\theta\cos 2\theta + \cos\theta\sin 2\theta = \sin 3\theta$ This can be awarded before the 3rd M1 Complete and convincing proof with no errors seen **SC1 : Substituting in π/3 to show $dy/d\theta$ = 0 can score full marks or M1 M1 A1 M1 M0 A0 if no other method shown**
8 (b)	Area enclosed by curve is $\frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\sin 2\theta\, d\theta = \frac{1}{2}a^2\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}a^2$ At P, $y = a\sqrt{\sin\left(\frac{2\pi}{3}\right)}\sin\left(\frac{\pi}{3}\right) = \frac{a\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}}$ At Q, $x = a\sqrt{\sin\left(\frac{2\pi}{6}\right)}\cos\left(\frac{\pi}{6}\right) = \frac{a\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}}$ So area of the square is $\frac{3a^2\sqrt{3}}{8}$ Hence area of shaded region is $\frac{3a^2\sqrt{3}}{8} - \frac{1}{2}a^2 = \frac{a^2}{8}\left(3\sqrt{3} - 4\right)$	M1* A1 B1 M1* A1 M1(dep*) A1 cao **[7]**	For using the correct area formula with the correct limits Correct area enclosed by the curve Correct exact distance of P and Q from the pole If they use decimals are used here, they can still recover the B1 later on if they continue to the correct exact area of the square Uses their distances to find the area of the square/their rectangle Correct exact area of the square Area of shaded region = area of rectangle – area enclosed by curve Obtains the correct area convincingly, giving the answer in the correct form ISW once correct form seen