Year 1/AS End of Topic Test: **Proof** | CM

ANSWER ALL QUESTIONS | TIME ALLOWED = 45 MINUTES

This test is marked out of 40 marks. You should show sufficient workings to make your methods clear. Answers without working may not gain full credit. Calculators are permitted in this test.

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1 The function f is defined such that

$$f(x) = 4x^2 - 24x + 39$$

(a) Show that the function f is positive for all x. **(4 marks)**

The function g is defined such that

$$g(x) = 4x^2 - 24x + c$$

where c is a constant.

(b) Prove that g is positive for all x if and only if $c > 36$. **(3 marks)**

2 Use proof by exhaustion to prove that the last digit of no square number is 8. **(3 marks)**

3 In each of the following cases, choose one of the following logic symbols

$$A \Rightarrow B \qquad A \Leftarrow B \qquad A \Leftrightarrow B$$

to describe the relationship between the statements A and B.

(a) For an integer n,

A: n is an even number B: n is a multiple of 4

(b) A: $x = 2$ B: $x^2 + x - 6 = 0$

(c) A: $x = 1$ B: $x^2 = 1$

(3 marks)

4 Alice claims that $(2x-3)^3 = 8x^3 - 27$. This is incorrect.

(a) Explain the mistake Alice has made. **(1 mark)**

(b) Use a counterexample to show that Alice's claim is false. **(2 marks)**

(c) Find the correct expansion of $(2x-3)^3$.

Give your answer in its simplest form. **(2 marks)**

5 Prove that if x and y are positive real numbers,

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

(4 marks)

6 If n is an integer that is not divisible by 3, show that $n^2 - 1$ is a multiple of 3. **(3 marks)**

7 Disprove each of the following statements using a suitable counterexample.

(a) $x^2 + 2x > 0$ for all x **(2 marks)**

(b) if two numbers have the same square, the two numbers are the same **(2 marks)**

8 Mark claims that "if two angles are complementary, they are congruent."

Find a pair of angles for which the statement is true and a pair for which it is false. **(4 marks)**

9 Let x and y be two distinct real numbers, where $x < y$.

(a) Explain why it follows that there exists a positive real number z such that $y - x = z$. **(1 mark)**

A theorem in mathematics can now be used to say that there exists a natural number n such that

$$n > \frac{1}{z}$$

(b) Use this theorem to show that

$$ny - nx > 1$$

explaining clearly where you have used the fact that z is positive. **(2 marks)**

(c) Deduce that

$$x < \frac{m}{n} < y$$

where m is an integer. **(2 marks)**

(d) Explain how parts (a) to (c) prove that between any two irrational numbers there exists a rational number. **(2 marks)**

END OF TEST