End of Topic Test Solutions

Notes:

- These solutions have been produced quickly and are intended to be brief.
- In most cases, alternative methods will not be shown. You may accept alternative methods provided the rubric of the question does not dictate that a specific approach that is required.
- In all cases, any abbreviations take their usual meanings.
- Grade boundaries are not provided for these worksheets.

These solutions are for the worksheet: **Proof**

Question 1:

(a)

$$4x^2 - 24x + 39 = 4(x^2 - 6x) + 39$$
$$= 4[(x-3)^2 - 9] + 39 \quad \text{(M1)}$$
$$= 4(x-3)^2 - 36 + 39 \quad \text{(M1)}$$
$$= 4(x-3)^2 + 3 \quad \text{(A1)}$$

$4(x-3)^2 \geq 0$ for all x, and so $4(x-3)^2 + 3 > 0$ for all x. (A1)

M1 – attempts to factorise out the 4 **AND** complete the square
M1 – re-distributes the 4
A1 – correct completing the square
A1 – conclusion

(b)

$$4(x^2 - 6x) + c = 4[(x-3)^2 - 9] + c \quad \text{(M1)}$$
$$= 4(x-3)^2 - 36 + c$$

Require $-36 + c > 0 \Leftrightarrow c > 36$ (M1)
So eq. is positive iff $c > 36$ (A1)

M1 – attempts to use their completed the square from (a)
M1 – identifies that you require $-36 + c > 0$
A1 – convincing proof + conclusion

Question 2:

Square numbers are formed those obtained by squaring a number.
All numbers end in 0 – 9 (inclusive), so a square number ends in any of the underlined digits:

(M1)

$0^2 = \underline{0}$
$1^2 = \underline{1}$
$2^2 = \underline{4}$
$3^2 = \underline{9}$
$4^2 = 1\underline{6}$
$5^2 = 2\underline{5}$
$6^2 = 3\underline{6}$
$7^2 = 4\underline{2}$
$8^2 = 6\underline{4}$
$9^2 = 8\underline{1}$

(M1)

None of the underlined digits are 8, so, by exhaustion, the last digit of no square number is 8.

(A1)

M1 – correct approach
M1 – test all numbers 0 – 9 (inclusive) and considers the last digit
A1 – correct conclusion with all square numbers correct

Question 3:

(a) $B \Rightarrow A$ (B1)

(b) $A \Rightarrow B$ (B1)

(c) $A \Rightarrow B$ (B1)

Question 4:

(a) Assumed that $(a-b)^3 = a^3 - b^3$ (B1 oe)

(b) Take $x = 1$, then (M1)

$\text{LHS} = (2-3)^2 = 1$

$\text{RHS} = 8(1)^3 - 27 = -19$

LHS $\neq$ RHS, so Alice's claim is false (A1)

M1 – chooses a counter-example (that works, i.e. not 0) **AND** **attempts** to show it does not work)
A1 – convincing conclusion

(c)

$$(2x-3)^3 = (2x-3)(2x-3)(2x-3)$$
$$= (4x^2 + ... + 9)(2x-3) \quad \text{(M1 A1)}$$
$$= 8x^3 - 36x^2 + 54x - 27$$

M1 – expands two brackets to form a 3TQ with $a = 4$ and $c = 9$ **OR** uses binomial expansion with coefficients 1, 3, 3, 1 and sum of powers for each term = 3
A1 – correct simplified expansion

Question 5

$(x-y)^2 \geq 0$ (M1)

$\Rightarrow x^2 - 2xy + y^2 \geq 0$

$\Rightarrow x^2 + y^2 \geq 2xy$ (A1)

$\Rightarrow \dfrac{x^2+y^2}{xy} \geq 2$ (M1)

$\Rightarrow \dfrac{x}{y} + \dfrac{y}{x} \geq 2$ (A1)

M1 – starts with correct inequality **AND** attempts to expand brackets
A1 – obtains correctly $x^2 + y^2 \geq 2xy$
dM1 – divides by xy
A1 – correct result, convincingly shown

{of course, accept a more technical solution that uses iff symbols}

ALTERNATIVE:

$\dfrac{x}{y} + \dfrac{y}{x} \geq 2$

$\Leftrightarrow \dfrac{x^2+y^2}{xy} \geq 2$

$\Leftrightarrow x^2 + y^2 \geq 2xy$

$\Leftrightarrow x^2 - 2xy + y^2 \geq 0$

$\Leftrightarrow (x-y)^2 \geq 0$

and this is true, therefore the original statement is true

This scores full marks for:
M1 – common denominator
A1 – correct expression (as above)
dM1 – attempts to factorise expression
A1 – convincing proof **AND** conclusion **AND** iff symbols.

**For a solution in the direction of the alternative that does not use iff symbols, i.e. just uses $\Rightarrow$ or no symbols at all, do not award the final A1. This is because they have shown *if* the statement is true, then $(x-y)^2 \geq 0$, which is not what the question has asked. **

Question 6:

If n is not a multiple of 3, then it is of the form (i) $n = 3k + 1$ or (ii) $n = 3k + 2$ (M1)

Case (i):

$$n^2 - 1 = (3k+1)^2 - 1$$
$$= 9k^2 + 6k$$
$$= 3(3k^2 + 2k)$$

Since it is of the form $3l$ for some integer l, $n^2 - 1$ is a multiple of 3 in case (i) (A1)

Case (ii)

$$n^2 - 1 = (3k+2)^2 - 1$$
$$= 9k^2 + 6k + 3$$
$$= 3(3k^2 + 2k + 1)$$

Since it is of the form $3m$ for some integer m, $n^2 - 1$ is a multiple of 3 in case (ii)

Therefore, by exhaustion, if n is not a multiple of 3, $n^2 - 1$ is a multiple of 3. (A1)

M1 – correct cases **AND** attempts to prove the statement is true in one of the cases
A1 – correct proof for one case (no conclusions needed)
A1 – correct proof for both cases with correct conclusions throughout

Question 7:

(a) Take $x = -1$, then (M1)

$$x^2 + 2x = (-1)^2 + 2(-1)$$
$$= -1 < 0$$

Therefore, the statement is not true for all x (A1)

M1 – chooses a correct counterexample **AND** attempts to show it doesn't work
A1 – correct illustration + conclusion

(b)
For example –1 and 1 have the same square, which is 1, but the numbers are not the same

(M1 A1)

M1 – chooses a correct counterexample **AND** attempts to show it doesn't work
A1 – correct illustration + conclusion

Question 8:

For the angles for which it works:

We want

$x + y = 90$

$x = y$

$2x = 90 \Rightarrow x = 45 \Rightarrow y = 45$

Therefore, it works if the two angles are 45° (M1 A1)

For the angles for which it does not work:

Choose, $x = 60$ and $y = 30$, which are complementary, as they add to 90. However, these aren't congruent.

(M1 A1)

M1 – attempts to find the angles for which the statement is true
A1 – works if both angles are 45°

M1 – chooses two complementary angles
A1 – explains that they are not congruent

Question 9 (Stretch and challenge):

(a) $x < y \Rightarrow y - x > 0$, therefore the value of $y - x$ must equal a positive real number, z. (B1)

B1 – for clear illustration that $y - x$ is positive.

(b)

$$n > \frac{1}{z}$$
$$\Rightarrow nz > 1 \qquad \text{(B1)}$$
$$\Rightarrow n(y-x) > 1$$
$$\Rightarrow ny - nx > 1$$

We have used the fact that z is positive to multiply the inequality by z in step 2 and not reverse the inequality sign (B1)

B1 – clear proof
B1 – clear indication where they have used the fact that z is positive. Indication of where is **enough** (i.e. no need to explain how)

(c)

By part (b), we know that there exists an integer m such that*

$$nx < m < ny \Rightarrow x < \frac{m}{n} < y \qquad \text{(M1 A1)}$$

M1 – deduces that there exists an m such that $nx < m < ny$
A1 – correct result

(d)
Since m and n are integers, m/n is rational. (B1)

x and y are real numbers, so in particular, they can be irrational.
Therefore, between any two irrational numbers, there is a rational number. (B1)

B1 – states or explains that m/n is rational
B1 – clear convincing conclusion

{*If you are struggling to understand why, think about it! Take some specific values and try to see why it works.}