AS

Mathematics

Pure Mathematics

for the **Edexcel** specification

FIRST EDITION

crashMATHS

Pure Mathematics

AS Level Mathematics Textbook

for the **Edexcel** Specification

First Edition

Written and published by crashMATHS.

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Zain Patel
Peter Hall

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CONTENTS

ABOUT THIS BOOK

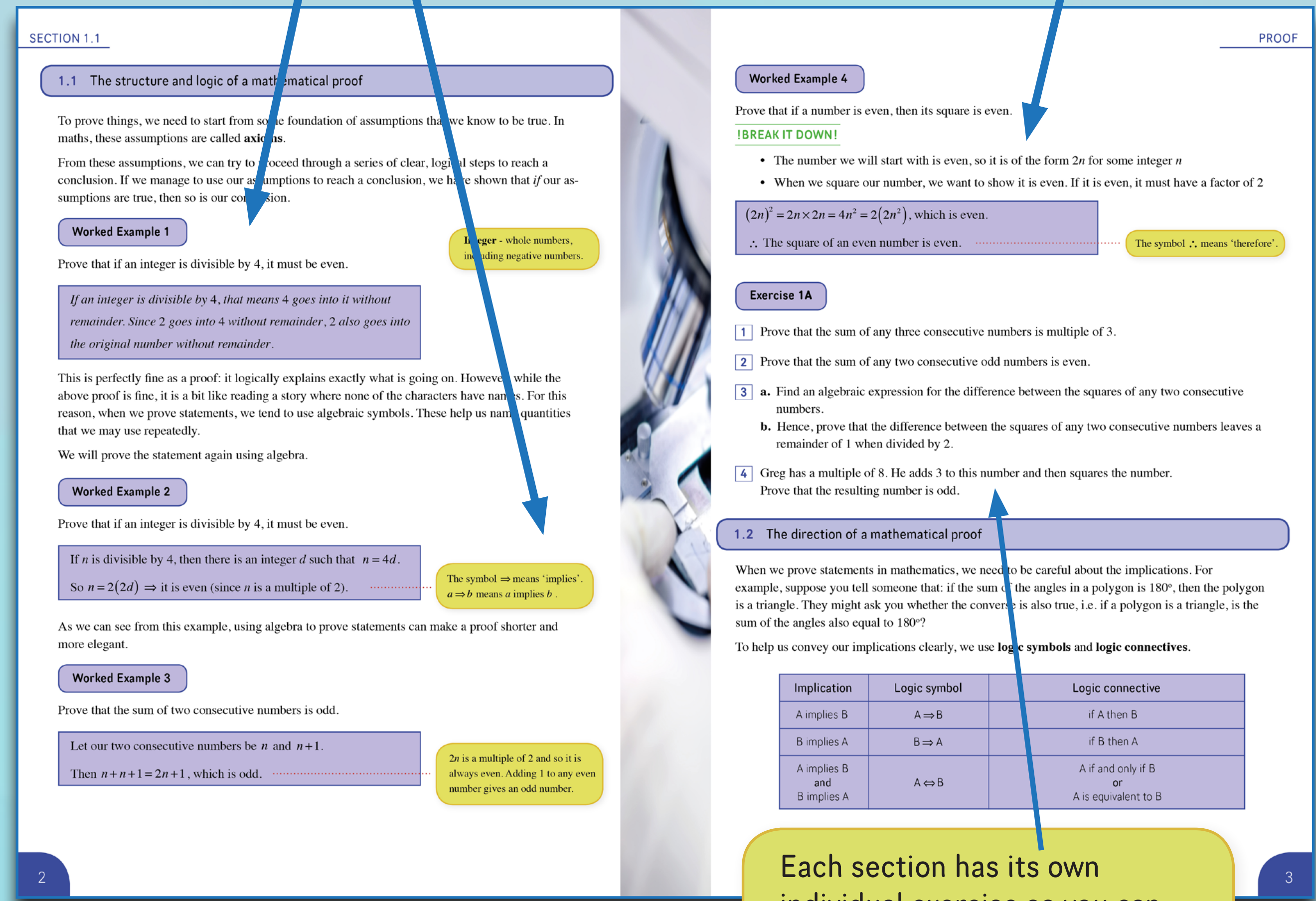
This book is designed to provide you with excellent preparation for your Edexcel AS Pure Mathematics examination:

- Written by subject experts
- Carefully designed for the Edexcel specification
- A focus on developing genuine understanding

How to use this coursebook

Plenty of worked examples to show you how to apply key ideas. Callouts are used to explain key stages in more detail.

Break-it-down sections are used in more difficult questions to show you how to tackle and approach the harder questions.



SECTION 1.1

1.1 The structure and logic of a mathematical proof

To prove things, we need to start from some foundation of assumptions that we know to be true. In maths, these assumptions are called **axioms**.

From these assumptions, we can try to proceed through a series of clear, logical steps to reach a conclusion. If we manage to use our assumptions to reach a conclusion, we have shown that *if* our assumptions are true, then so is our conclusion.

Worked Example 1

Prove that if an integer is divisible by 4, it must be even.

If an integer is divisible by 4, that means 4 goes into it without remainder. Since 2 goes into 4 without remainder, 2 also goes into the original number without remainder.

Integer - whole numbers, including negative numbers.

This is perfectly fine as a proof: it logically explains exactly what is going on. However, while the above proof is fine, it is a bit like reading a story where none of the characters have names. For this reason, when we prove statements, we tend to use algebraic symbols. These help us name quantities that we may use repeatedly.

We will prove the statement again using algebra.

Worked Example 2

Prove that if an integer is divisible by 4, it must be even.

If n is divisible by 4, then there is an integer d such that $n = 4d$.

So $n = 2(2d) \Rightarrow$ it is even (since n is a multiple of 2).

The symbol $\Rightarrow$ means 'implies'. $a \Rightarrow b$ means a implies b.

As we can see from this example, using algebra to prove statements can make a proof shorter and more elegant.

Worked Example 3

Prove that the sum of two consecutive numbers is odd.

Let our two consecutive numbers be n and $n+1$.

Then $n+n+1 = 2n+1$, which is odd.

$2n$ is a multiple of 2 and so it is always even. Adding 1 to any even number gives an odd number.

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PROOF

Worked Example 4

Prove that if a number is even, then its square is even.

!BREAK IT DOWN!

- The number we will start with is even, so it is of the form $2n$ for some integer n
- When we square our number, we want to show it is even. If it is even, it must have a factor of 2

$(2n)^2 = 2n \times 2n = 4n^2 = 2(2n^2)$, which is even.

$\therefore$ The square of an even number is even.

The symbol $\therefore$ means 'therefore'.

Exercise 1A

1. Prove that the sum of any three consecutive numbers is multiple of 3.
2. Prove that the sum of any two consecutive odd numbers is even.
3. a. Find an algebraic expression for the difference between the squares of any two consecutive numbers.
   b. Hence, prove that the difference between the squares of any two consecutive numbers leaves a remainder of 1 when divided by 2.
4. Greg has a multiple of 8. He adds 3 to this number and then squares the number. Prove that the resulting number is odd.

1.2 The direction of a mathematical proof

When we prove statements in mathematics, we need to be careful about the implications. For example, suppose you tell someone that: if the sum of the angles in a polygon is 180°, then the polygon is a triangle. They might ask you whether the converse is also true, i.e. if a polygon is a triangle, is the sum of the angles also equal to 180°?

To help us convey our implications clearly, we use **logic symbols** and **logic connectives**.

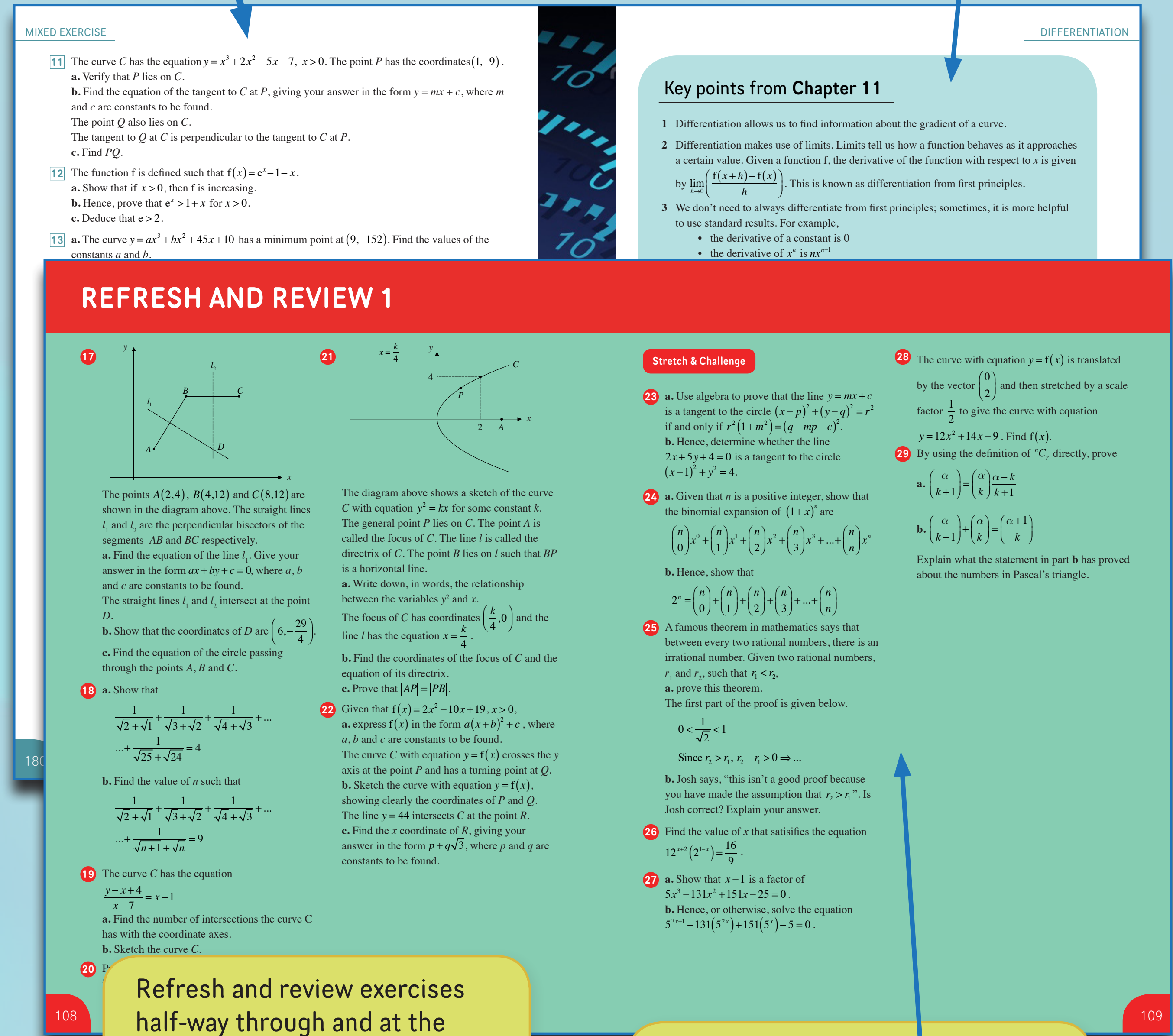
Implication	Logic symbol	Logic connective
A implies B	$A \Rightarrow B$	if A then B
B implies A	$B \Rightarrow A$	if B then A
A implies B and B implies A	$A \Leftrightarrow B$	A if and only if B or A is equivalent to B

3

Each section has its own individual exercise so you can have plenty of practice with the key ideas before moving on.

Each chapter ends with a mixed exericse with plenty of questions to consolidate what you have learnt.

Chapters conclude with a summary of key points. These briefly outline the key ideas from the chapter to help you refresh your memory before moving on.


MIXED EXERCISE

11 The curve C has the equation $y = x^3 + 2x^2 - 5x - 7$, $x > 0$. The point P has the coordinates $(1, -9)$.
a. Verify that P lies on C.
b. Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants to be found.
The point Q also lies on C.
The tangent to Q at C is perpendicular to the tangent to C at P.
c. Find PQ.

12 The function f is defined such that $f(x) = e^x - 1 - x$.
a. Show that if $x > 0$, then f is increasing.
b. Hence, prove that $e^x > 1 + x$ for $x > 0$.
c. Deduce that $e > 2$.

13 **a.** The curve $y = ax^3 + bx^2 + 45x + 10$ has a minimum point at $(9, -152)$. Find the values of the constants a and b.

DIFFERENTIATION

Key points from **Chapter 11**

1 Differentiation allows us to find information about the gradient of a curve.

2 Differentiation makes use of limits. Limits tell us how a function behaves as it approaches a certain value. Given a function f, the derivative of the function with respect to x is given by $\lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$. This is known as differentiation from first principles.

3 We don't need to always differentiate from first principles; sometimes, it is more helpful to use standard results. For example,
- the derivative of a constant is 0
- the derivative of x^n is nx^{n-1}

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REFRESH AND REVIEW 1

17 The points $A(2,4)$, $B(4,12)$ and $C(8,12)$ are shown in the diagram above. The straight lines l_1 and l_2 are the perpendicular bisectors of the segments AB and BC respectively.
a. Find the equation of the line l_1. Give your answer in the form $ax + by + c = 0$, where a, b and c are constants to be found.
The straight lines l_1 and l_2 intersect at the point D.
b. Show that the coordinates of D are $\left(6, -\frac{29}{4}\right)$.
c. Find the equation of the circle passing through the points A, B and C.

18 **a.** Show that
$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \dots + \frac{1}{\sqrt{25}+\sqrt{24}} = 4$$
b. Find the value of n such that
$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \dots + \frac{1}{\sqrt{n+1}+\sqrt{n}} = 9$$

19 The curve C has the equation
$$\frac{y - x + 4}{x - 7} = x - 1$$
a. Find the number of intersections the curve C has with the coordinate axes.
b. Sketch the curve C.

20 P

21 The diagram above shows a sketch of the curve C with equation $y^2 = kx$ for some constant k.
The general point P lies on C. The point A is called the focus of C. The line l is called the directrix of C. The point B lies on l such that BP is a horizontal line.
a. Write down, in words, the relationship between the variables y^2 and x.
The focus of C has coordinates $\left(\frac{k}{4}, 0\right)$ and the line l has the equation $x = \frac{k}{4}$.
b. Find the coordinates of the focus of C and the equation of its directrix.
c. Prove that $|AP| = |PB|$.

22 Given that $f(x) = 2x^2 - 10x + 19$, $x > 0$,
a. express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants to be found.
The curve C with equation $y = f(x)$ crosses the y axis at the point P and has a turning point at Q.
b. Sketch the curve with equation $y = f(x)$, showing clearly the coordinates of P and Q.
The line $y = 44$ intersects C at the point R.
c. Find the x coordinate of R, giving your answer in the form $p + q\sqrt{3}$, where p and q are constants to be found.

Stretch & Challenge

23 **a.** Use algebra to prove that the line $y = mx + c$ is a tangent to the circle $(x-p)^2 + (y-q)^2 = r^2$ if and only if $r^2(1+m^2) = (q - mp - c)^2$.
b. Hence, determine whether the line $2x + 5y + 4 = 0$ is a tangent to the circle $(x-1)^2 + y^2 = 4$.

24 **a.** Given that n is a positive integer, show that the binomial expansion of $(1+x)^n$ are
$$\binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$$
b. Hence, show that
$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

25 A famous theorem in mathematics says that between every two rational numbers, there is an irrational number. Given two rational numbers, r_1 and r_2, such that $r_1 < r_2$,
a. prove this theorem.
The first part of the proof is given below.
$$0 < \frac{1}{\sqrt{2}} < 1$$
Since $r_2 > r_1$, $r_2 - r_1 > 0 \Rightarrow \dots$
b. Josh says, "this isn't a good proof because you have made the assumption that $r_2 > r_1$". Is Josh correct? Explain your answer.

26 Find the value of x that satisifies the equation $12^{x+2}(2^{1-x}) = \frac{16}{9}$.

27 **a.** Show that $x - 1$ is a factor of $5x^3 - 131x^2 + 151x - 25 = 0$.
b. Hence, or otherwise, solve the equation $5^{3x+1} - 131(5^{2x}) + 151(5^x) - 5 = 0$.

28 The curve with equation $y = f(x)$ is translated by the vector $\binom{0}{2}$ and then stretched by a scale factor $\frac{1}{2}$ to give the curve with equation $y = 12x^2 + 14x - 9$. Find $f(x)$.

29 By using the definition of ${}^nC_r$ directly, prove
a. $\binom{\alpha}{k+1} = \binom{\alpha}{k}\frac{\alpha - k}{k+1}$
b. $\binom{\alpha}{k-1} + \binom{\alpha}{k} = \binom{\alpha+1}{k}$
Explain what the statement in part **b** has proved about the numbers in Pascal's triangle.

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Refresh and review exercises half-way through and at the end of the book to test many topics at once.

Each refresh and review exercise ends with a selection of stretch and challenge questions to help develop your understanding even further.

Chapter 1: Proof

Learning goals for Chapter 1

By the end of this chapter, you should be able to:

- understand the essence of a mathematical proof
- prove mathematical statements using proof by deduction
- prove mathematical statements using proof by exhaustion
- disprove mathematical statements by counterexample

Prior Knowledge

This chapter will assume a basic understanding of algebra and we may mention some concepts you'll have met at GCSE to add context to some of the ideas. The algebra that we will use is explained in detail in Chapter 2, but you are likely to have met these ideas already.

In mathematics, we often want to prove things. This is different from most other fields, where we are usually content with convincing ourselves beyond reasonable doubt, such as in the sciences. A mathematical proof doesn't just establish something beyond reasonable doubt but ensures that it is 100% true. But this isn't the only the reason we try to prove things: a proof should also explain why a result is true and give us further insight.

1.1 The structure and logic of a mathematical proof

To prove things, we need to start from some foundation of assumptions that we know to be true. In maths, these assumptions are called **axioms**.

From these assumptions, we can try to proceed through a series of clear, logical steps to reach a conclusion. If we manage to use our assumptions to reach a conclusion, we have shown that *if* our assumptions are true, then so is our conclusion.

Worked Example 1

Integer - whole numbers, including negative numbers.

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As we can see from this example, using algebra to prove statements can make a proof shorter and more elegant.

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Then $n+n+1 = 2n+1$, which is odd.

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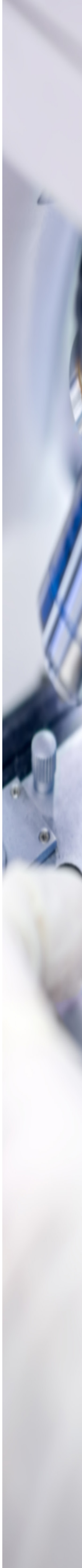

Worked Example 4

Prove that if a number is even, then its square is even.

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- The number we will start with is even, so it is of the form $2n$ for some integer n
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$(2n)^2 = 2n \times 2n = 4n^2 = 2(2n^2)$, which is even.

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Exercise 1A

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2 Prove that the sum of any two consecutive odd numbers is even.

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4 Greg has a multiple of 8. He adds 3 to this number and then squares the number.
Prove that the resulting number is odd.

1.2 The direction of a mathematical proof

When we prove statements in mathematics, we need to be careful about the implications. For example, suppose you tell someone that: if the sum of the angles in a polygon is 180°, then the polygon is a triangle. They might ask you whether the converse is also true, i.e. if a polygon is a triangle, is the sum of the angles also equal to 180°?

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In Worked Example 1, we proved that if an integer is divisible by 4, it must be even. But we know that if an integer is even, it need not be divisible by 4. This is not an if and only if statement.

On the other hand, a polygon is a triangle if and only if the sum of the angles in the polygon is 180º. The two statements - ‘a polygon is a triangle’ and ‘the sum of the angles in the polygon is 180º’ - are equivalent statements. When we want to prove two statements are equivalent, we need to prove **both directions**.

Worked Example 5

Prove that a number is even if and only if its square is even.

!BREAK IT DOWN!

- This is an if and only if proof, so we need to prove both directions.
- We proved the first direction in Worked Example 4. Now we have to prove the other direction.

First direction: if a number is even, then its square is even:

$(2n)^2 = 2n \times 2n = 4n^2 = 2(2n^2)$, which is even.

From Worked Example 4.

$\therefore$ If a number is even, its square is even

Second direction: if the square of a number is even, the number is even:

$n^2 = 2k$ for some integer k

Rewrite the assumption using algebra.

As 2 divides the right-hand side, it divides the left-hand side.

$\Rightarrow$ 2 divides one (or both) of the factors on the left-hand side.

These are both n, so 2 divides n.

$\therefore$ If the square of a number is even, the number is even

$\therefore$ a number is even $\Leftrightarrow$ its square is even

We have proved both directions, so we can conclude that they are equivalent.

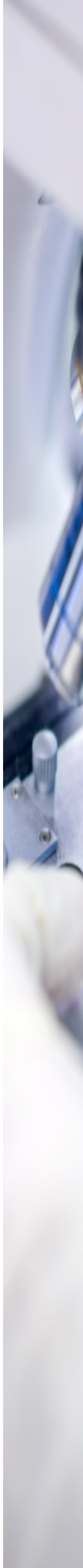
In a lot of the maths we do, we deal with equivalent statements. We don’t always have to show both directions when the equivalence is obvious. Equivalences that follow naturally don’t usually need to be proved explicitly - we can just use logic symbols to indicate them.

Worked Example 6

Prove $x+3=4$ if and only if $x=1$.

$x+3=4 \Leftrightarrow x=4-3=1$

This equivalence follows quite naturally - we can ‘see’ both directions hold without much explanation, so the logic symbol alone will do.



Exercise 1B

1 Connect each of the following statements with the relevant logic symbol. Explain your answer in each case (full proofs are not required).

a. **Statement A:** Alice has stripes
Statement B: Alice is a tiger

b. **Statement A:** x is a square number
Statement B: $x = 1$

c. **Statement A:** the angles on ABC add to 180°
Statement B: ABC is a straight line

d. **Statement A:** triangle T has three equal sides
Statement B: triangle T has three equal angles

2 Prove that n is odd if and only if n^2 is odd.

3 Alex has four statements: A, B, C and D.
How could Alex prove that the statements are equivalent?

4 Show that the statements
i. n is even and **ii.** $7n+4$ is even
are equivalent.

5 Suppose that x and y are distinct integers. Prove that $(x+1)^2 = (y+1)^2$ if and only if $x+y=-2$.

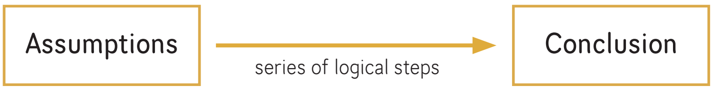
6 If a, b, c and d are real numbers, with b and d non-zero, prove that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

Real numbers - any number you can find on a number line is real.

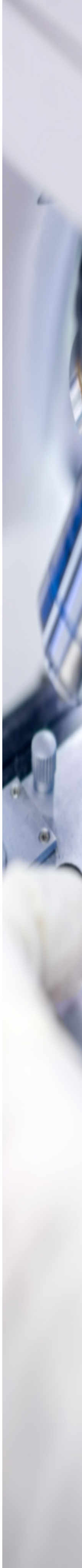
1.3 Proof by deduction

Mathematicians have many different methods of proof. Some statements may be difficult to prove using one method, while others may be well suited to that method.

Proof by deduction is one technique for proving things. It is one of the most common methods of proof and is widely used. It is the method of proof we have been using so far. And like we have seen, it often consists of using algebra to assist the logic behind the proof.



Proof by deduction involves beginning with a set of assumptions, then through a series of logical steps, we can *deduce* a conclusion.



Worked Example 7

Prove that if $x > 2$ and $y > 1$, then $x + y > 3$.

Link: Inequalities CH3

Adding the two inequalities together gives $x + y > 2 + 1 = 3$

Worked Example 8

Show that $a^2 + b^2 \geq 2ab$ for all values of a and b.

For all a and b, we know that $(a-b)^2 \geq 0$

Any square number is greater than or equal to 0.

Expanding the brackets gives $a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$

Worked Example 9

Prove that $n^2 + 4n + 5 > 0$ for all n.

Completing the square gives $n^2 + 4n + 5 = (n+2)^2 + 1$.

Link: Quadratics CH2

Since $(n+2)^2 \geq 0$, $(n+2)^2 + 1 \geq 1$ and 1 is always greater than 0.

$\therefore\ n^2 + 4n + 5 > 0$ for all n.

Exercise 1C

1 Suppose that: A implies B, B implies C and C implies D .
Prove that if A is true, then so is D.

2 Prove that the sum of any two even numbers is even.

3 Using the inequality from Worked Example 8, choose suitable values of a and b to show that

a. $\frac{1}{2}(x+y) \geq \sqrt{xy}$, $x, y > 0$ **b.** $4^x + 4^{-x} \geq 2$

You may want to try **Q3** after completing **CH2**

1.4 Proof by exhaustion

Proof by exhaustion is another method of proof. It involves trying all of the options.

This type of proof is not very practical where there are lots of options, because it can take a lot of time to try every option. However, when mathematicians have to prove a difficult theorem, they may sometimes break it down into a finite number of different options/cases and prove these separately using another type of proof.

Worked Example 10

Let n be an integer. Prove by exhaustion that $n^2 - 5n + 4$ is positive for $6 \leq n \leq 8$.

We try all of the options:
$n = 6$: $(6)^2 - 5(6) + 4 = 10$, which is positive.
$n = 7$: $(7)^2 - 5(7) + 4 = 18$, which is positive.
$n = 8$: $(8)^2 - 5(8) + 4 = 28$, which is positive.
So, by exhaustion, $n^2 - 5n + 4$ is positive for $6 \leq n \leq 8$.

Worked Example 11

Prove by exhaustion that if n is not divisible by 3, then $n^2 = 3k + 1$ for some integer k.

!BREAK IT DOWN!

- We are going to use exhaustion, so we need to think about our different options for n.
- If n is not divisible by 3, then it leaves a remainder of 1 or 2, so can be written as either

  1. $n = 3r + 1$ or **2.** $n = 3r + 2$

Case **1**: $n = 3r + 1 \Rightarrow n^2 = (3r+1)^2 = 9r^2 + 6r + 1$

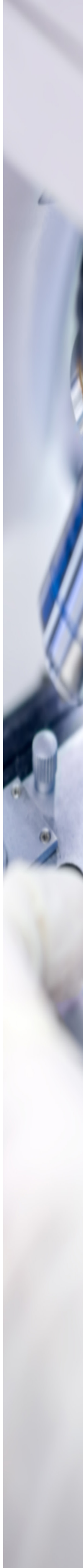
So, we have $n^2 = 3(3r^2 + 2r) + 1$, and so the statement is true for Case **1**. $k = 3r^2 + 2r$

Case **2**: $n = 3r + 2 \Rightarrow n^2 = (3r+2)^2$

$= 9r^2 + 12r + 4$

$= 9r^2 + 12r + 3 + 1$

$= 3(3r^2 + 4r + 1) + 1$, so it is also true for Case **2**. $k = 3r^2 + 4r + 1$



Exercise 1D

1. If x is a positive integer less than 5, prove that the last digit of x^5 is x.
2. Prove, for integers between 10 and 40, that reversing the digits of a multiple of 3 gives a number that is also a multiple of 3.
3. Prove, for the first 12 positive multiples of 9, the sum of the digits of a multiple of 9 is a multiple of 3.
4. Let x be an integer in the interval $3 \leq x \leq 6$.
   Using exhaustion, prove that $-20 < x^3 - 8x^2 + 10x + 6 < 0$.
5. Prove that every integer between 20 and 30 can be written as the sum of five square numbers.
6. If a is not a multiple of 3, use exhaustion to prove that $a^2 - 1$ is a multiple of 3.

1.5 Disproof by counter example

Disproof by counter example is a method mathematicians often use to disprove statements. It involves giving one example that contradicts the initial statement. An example that contradicts a statement is known as a *counter example*.

The fact that we only need one example to disprove any statement may seem counter-intuitive. So to help appreciate this, imagine your friend has a basket full of apples. Your friend then tells you that all of the apples in his basket are red apples. To his surprise, you pull out a green apple from his basket. This green apple in his basket is a counter example to his statement; he might have many red apples in his basket, but your counter example proved that not all of them are.

Worked Example 12

Use a counter example to show that the statement $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for all a and b is false.

For example, take $a = 1$ and $b = 3$.

Then, $\sqrt{a+b} = \sqrt{1+3} = \sqrt{4} = 2$

But $\sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{3} = 2.73... \neq 2$

So $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ for all a and b.

The symbol $\neq$ stands for ‘not equal to’.

It is important to remember that a counter example doesn’t say that a statement is never true. It just shows that at least some part of the statement is not *always* true for the conditions given.

Exercise 1E

1. James says that "all primes are odd". Is he correct?

2. Use a suitable counter example to disprove the following statements
   a. the sum of two distinct square numbers is a square number
   b. all positive cube numbers are either even or one less than a multiple of 3
   c. if the sum of two integers is even, then one of the summands is even
   d. all natural numbers are either prime or have more than one factor
   e. if a and b are natural numbers, then so is their difference

Natural numbers - the counting numbers $\{1, 2, 3, ...\}$.

3. Show that if $x > y$, it is not necessarily true that $ax > ay$ for all values of a.

4. Disprove the claim that $x^2 - 8x + 6$ is negative for all positive integers x.

5. Disprove the statement that every positive integer is divisible by some prime.

Mixed Exercise

1. Prove that the sum of two consecutive multiples of 3 is a multiple of 3.

2. **a.** Given that a divides b, write down an expression for a in terms of b.
   b. Prove that if a divides b and b divides c, then a divides c.

3. Prove or disprove that if $x > y$, then $x^2 > y^2$, for all values of x and y.

4. **a.** Use exhaustion to prove that the sum of two positive integers between 5 and 10 is larger than the individual summands.
   b. Use a suitable counter example to prove that the sum of two integers is not necessarily larger than the individuals summands.

5. Show that $x^2 + 3x + 4 > 0$ for all values of x.

6. Prove or disprove that for all a and b, $a^2 = b^2 \Rightarrow a = b$.

7. **a.** Prove by exhaustion that if p is a prime less than 7, then $2p + 1$ is also prime.
   b. Prove that the statement in part **a** is not true for all prime numbers p.

8. Prove that all natural numbers greater than 1 are divisible by some prime.

9. Alex claims that $n^2 + 3n + 2$ is a prime number for all even values of n.
   Show that Alex's claim is untrue.

10. Prove that if n is an integer, $3n^2 + n + 14$ is even.

11. Josh wants to create a fraction with numerator 1. He says, "I can place any integer in the denominator of my fraction to create my fraction." Is Josh's claim true?

12 **a.** "If n is an integer and n^2 is divisible by 4, then n is divisible by 4". Prove or disprove this claim.

b. "If n is an integer and n is divisible by 4, then n^2 is divisible by 4". Prove or disprove this claim.

c. Alex proposes: n is divisible by 4 $\Leftrightarrow$ n^2 is divisible by 4

Is Alex correct? Justify your answer.

13 Is it true that $x=1$ if and only if $x^2=1$? Explain your answer using a proof or a suitable counterexample.

14 **a.** Prove that $a^2+b^2 \geq 2ab$ for all a and b.

b. Hence, show that for $\varepsilon > 0$,

i. $ab \leq \frac{1}{2}\left(\varepsilon a^2 + \frac{b^2}{\varepsilon}\right)$ **ii.** $\frac{x^2}{x^4+\varepsilon} \leq \frac{1}{2\sqrt{\varepsilon}}$

15 Suppose n is not divisible by 5.

Prove that when n^2 is divided by 5, there is a remainder of 1 or a remainder of 4.

Key points from **Chapter 1**

1 There are various methods mathematicians can use to prove statements.

2 Direction in a mathematical proof is important:

- if $A \Rightarrow B$, then if A happens, so does B.
- if $A \Leftarrow B$, then if B happens, so does A.
- if $A \Leftrightarrow B$, then A happens if and only if B happens. A and B are said to be equivalent.

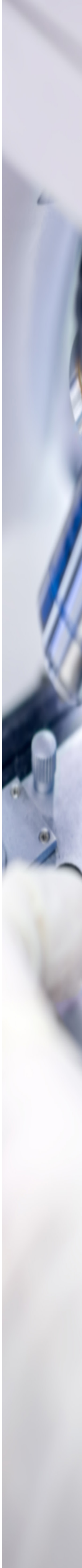
3 Proof by deduction involves starting with a set of assumptions and, from these assumptions, following a series of logical steps through to a conclusion.

4 When proving things, it is usually simpler to use algebra.

4 If we can break a statement down into several cases/options, it may be easier to consider each case separately. This type of proof is called proof by exhaustion.

5 It is usually not enough to prove a general statement by checking that is valid for a few values.

6 It is enough to disprove a statement by giving a single counter example.



2.2 Laws of indices

We have already used some basic laws of indices in previous exercises and examples. Now we will look at them directly and build our understanding of them.

Laws of indices
$a^m \times a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$\left(a^m\right)^n = a^{mn}$
$(ab)^m = a^m b^m$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
$a^0 = 1$
$a^{-m} = \frac{1}{a^m}$
$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Worked Example 21

Simplify $p^4 \times p^5 \times p^3 \times q^3$.

$$p^4 \times p^5 \times p^3 \times q^3 = p^{12} q^3$$

We can combine the powers of the terms of p together. Notice we cannot combine the powers of the p terms with the q term, because they are not the same **base**.

Worked Example 22

Simplify $\left(x^{-1}y^{-2}\right)^2$.

$$\left(x^{-1}y^{-2}\right)^2 = x^{-2}y^{-4}$$

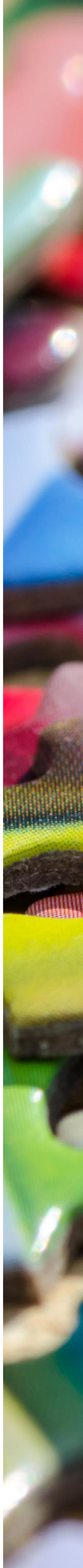
We can also write this as $\frac{1}{x^2y^4}$ or $\left(\frac{1}{xy^2}\right)^2$.

Worked Example 23

Express $\frac{1}{\sqrt{2}} \times \left(2^{-4}\right)^2$ in the form $\frac{1}{\sqrt{2^a}}$, where a is a constant to be found.

$$\frac{1}{\sqrt{2}} \times \left(2^{-4}\right)^2 = \frac{1}{2^{\frac{1}{2}}} \times 2^{-8} = \frac{1}{2^{\frac{1}{2}}} \times \frac{1}{2^8} = \frac{1}{2^{\frac{17}{2}}} = \frac{1}{\sqrt{2^{17}}}$$

Write all the terms in index form and combine them using the laws. In this case, $a = 17$.



Worked Example 24

Solve the equation $2^{x+2} = 4^x$.

$$\boxed{2^{x+2} = 2^{2x} \Leftrightarrow x + 2 = 2x \Leftrightarrow x = 2}$$

If we write 4 as 2^2, we can compare the powers on both sides, because the only way for $2^x = 2^y$ is if $x = y$.

Exercise 2F

1 Without using a calculator, find the values of

a. $2^4 \times 2^2$ **b.** $3^0 \times 3^6 \times 3^{-3}$

c. $2^4 \times 3^1 \times 2^2 \times 3^2$ **d.** $2^3 \times 4^{-2} \times 5^3 \times 125^{-1}$

e. $\sqrt{5^2} \times 25^2 \div 5^2$ **f.** $\sqrt{2^4 \times 2^{-3} \times 4^{-1}}$

g. $2^3 \times 4^{-2} \times 2^8$ **h.** $\dfrac{4^4}{2^{12}}$

i. $\dfrac{3^4 \times 2^7}{2^7 \times 4^2 \times 3^5}$ **j.** $\dfrac{25^3 \times 125^{-1}}{5^4 \times 5^{-2}}$

k. $\sqrt{8} \times \sqrt[3]{3} \times \sqrt[3]{9} \times 2^{3/2}$ **l.** $\dfrac{\sqrt[5]{32^3} \times 3^2}{\sqrt{2^{-2}}}$

2 Simplify the following expressions

a. $\dfrac{a^7}{a^3 \times a^2}$ **b.** $\dfrac{p^2}{p^{-2}}$

c. $\dfrac{a^2 \times b^7}{a^{-1} \times b^4}$ **d.** $\dfrac{u^2 \times v^4 \times w^{-5}}{u^{-1} \times v^7 \times w^3}$

e. $\dfrac{a^b \times a^{2b}}{a^{3b} \times a^{-4b}}$ **f.** $(2m)^3 \times (4m)^2$

g. $\dfrac{(3x)^3 \times (2y)^{-1}}{4y^{-2} \times x^2}$ **h.** $\dfrac{(a^2b)^3 \times (c^3d^2)^2}{c^2 \times d^5}$

i. $\sqrt{a^3} \times \sqrt[4]{a^6}$ **j.** $\sqrt{4x} \times \sqrt[3]{27y} \times \sqrt{x^3} \times \sqrt[9]{y^6}$

3 Solve the following index equations

a. $2^{5x+4} = 2^{3x+6}$ **b.** $2^{7x+3} = 4^{2x+6}$

c. $3^{10x+1} = 3^{3x+4}$ **d.** $3^{5x+4} = \left(\sqrt{3}\right)^{4x+10}$

e. $25^{9x+3} = 125^{3x+5}$ **f.** $3^{2x+1} = 3^{4-6x}$

g. $2^{4p+3} \times 4^{3p+2} = 16^{2p+8}$ **h.** $6^{2x+5} = 36^{x^2}$

4 Given that $\left(\sqrt{3}\right)^{4x+6} \times 27^{y+4} = \sqrt[4]{9^{2y+4}} \times 3^x$, express y in terms of x.

Learning goals for Chapter 3

By the end of this chapter, you should be able to:

- solve simple simultaneous equations in two variables
- understand the notation used for inequalities
- solve linear and quadratic inequalities
- interpret the solutions to various inequalities

Prior Knowledge

For this chapter, you will need to be confident with the content in Chapters 1 and 2. In particular, you need to be confident with solving both linear and quadratic equations. The inequalities section of the chapter will assume knowledge of the shape of a quadratic curve.

Chapter 3:
Equations and Inequalities

This chapter will give you an introduction to systems of equations and inequalities. We will consider some simple cases, where our system consists of only two equations. In computer science, systems of equations are incredibly important as they can be used to encode large quantities of data.

3.1 Solving simple simultaneous equations by elimination

You have been solving equations with one unknown variable for many years. We will now extend this. Equations that have more than one unknown can have an infinite number of solutions.

For example, the equation $x+2y=10$ has many solutions (can you think of some?). However, to solve this equation uniquely, another equation needs to be used alongside it. That is the essence of what we will be doing for the first half of this chapter: looking at systems of two equations in two variables and the methods that we can use to solve them.

Worked Example 1

Solve the simultaneous equations $x+2y=10$ and $x+y=8$.

!BREAK IT DOWN!

- We want to find the value of x and the value of y that solves both equations. Our method is going to be to try to eliminate one of the variables using both of these equations so that we end up with an equation in terms of just x or y.

$x+2y=10 \quad \text{(Eq. 1)}$

$x+y=8 \quad \text{(Eq. 2)}$

Subtracting (2) from (1) gives $y=2$

Substitute this into (2) to find x:

$x+2=8 \Rightarrow x=6$

Therefore, the solution is $x=6$, $y=2$.

Label the equations.

We notice that if we subtract the equations, we get a simple equation just in terms of y.

Substitute the value of y into any of the equations to find the corresponding value of x.

Check your answer by substituting the values into the original equations.

Worked Example 2

Solve the simultaneous equations $2x+3y=8$ and $4x-3y=4$.

$2x+3y=8 \quad \text{(Eq. 1)}$

$4x-3y=4 \quad \text{(Eq. 2)}$

Adding (1) and (2) gives: $6x=12 \Rightarrow x=2$

Substitute this into (1) to find y:

$2(2)+3y=8 \Rightarrow 3y=4 \Rightarrow y=\frac{4}{3}$

Therefore, the solution is $x=2$, $y=\frac{4}{3}$.

Label the equations.

We notice that if we add the equations, we get a simple equation just in terms of x.

Substitute the value of x into any of the equations to find the corresponding value of y.

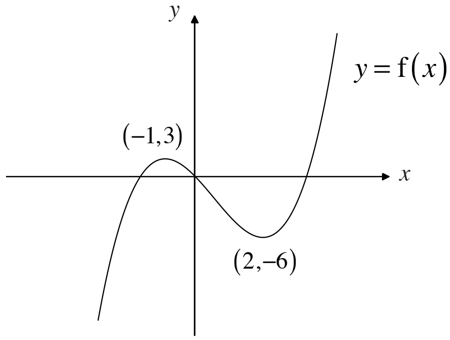
Check your answer by substituting the values into the original equations.

Mixed Exercise

1 The curve C has the equation $y=x^3+2x^2+x$.
a. Sketch the graph of C.
b. Hence, or otherwise, sketch on a separate axis, the graph of $y=(x+2)^3+2(x+2)^2+(x+2)$.
On each sketch, you should show clearly the coordinates of any points at which the curve crosses or meets the coordinate axes.

2 **a.** Prove that the curve with equation $y=x^2+5x+10$ has no real roots.
b. Find the minimum value of the curve $y=x^2+5x+10$.
c. Hence, sketch the curve with equation $y=x^2+5x+10$.
On your sketch, you should show clearly the coordinates of any points at which the curve crosses or meets the coordinate axes.
d. Describe geometrically the single transformation that maps the curve with equation $y=x^2$ onto the curve with equation $y=x^2+5x+10$.

3 The diagram shows a sketch of the curve with equation $y=\mathrm{f}(x)$. The curve has a maximum point $(-1,3)$ and a minimum point at $(2,-6)$. On separate axes, sketch the curves with equation
a. $y=\mathrm{f}(2x)$
b. $y=\mathrm{f}(x)-4$
c. $y=-\mathrm{f}(x)$
On each sketch, you should show clearly the coordinates of the minimum and maximum point.



4 The curve $y=x^2+kx+2$ intersects the line $y=2x+1$ twice.
Find the range of possible values of k.

5 On separate axes, sketch the curves with equation
a. $y=x^{-1}$
b. $y=x^{-1}-4$
c. $y=(x+1)^{-1}$
On each sketch, you should show clearly the coordinates of any points at which the curves cross or meet the coordinate axes. In each case, you should state the equations of any asymptotes.

6 Show, by sketching two suitable equations on the same axes, that the equation

$$(x+1)(x-3)^2=\frac{1}{x-2}-3$$

has only two solutions. On each sketch, you should also show clearly the coordinates of any points at which the curves cross or meet the coordinate axes. In each case, you should state the equations of any asymptotes.

7 **a.** Sketch, on the same axes, the straight line $y-4x=5$ and the curve $y=x^2(3-x)$.
On your sketch, you should show clearly the coordinates of any points at which the curves cross or meet the coordinate axes.
b. State, with reason, the number of solutions to the equation $x^3-3x^2+4x+5=0$.

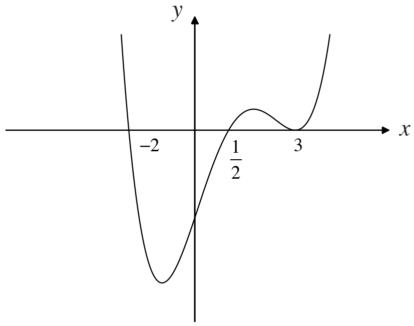
8 Show that the curve with equation $\dfrac{3x+y-21}{x-5}=x+3$ has three intersections with the coordinate axes.

9 **a.** Use algebra to show that $\dfrac{2x+3}{x-2}=a+\dfrac{b}{x-2}$, where a and b are integers to be found.
b. Find the coordinates at which the curve with equation $y=\dfrac{2x+3}{x-2}$ meets the coordinate axes.
c. Sketch the graph of the curve $y=\dfrac{2x+3}{x-2}$, stating the equations of any asymptotes.

10 Find the values of k for which the line $y=2x+k$ is a tangent to the circle $x^2+(y-2)^2=4$.

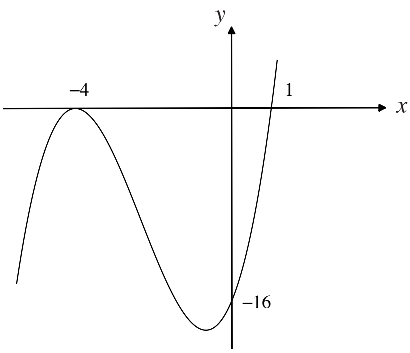
11 **a.** Solve the simultaneous equations $4y=12x-9$ and $y=x^2$.
b. Write down a geometric relationship between the line $4y=12x-9$ and the curve $y=x^2$.

12 The diagram shows a sketch of the curve of the function f.
Sketch, on separate axes, the curves with equation
a. $y=\mathrm{f}(x-2)$
b. $y=\mathrm{f}(-x)$
c. $y=3\mathrm{f}(x)$
On each sketch, you should show clearly the coordinates of any points at which the curve meet or cross the coordinate axes.



13 A curve C has equation $y=x^2+(2k-1)x+8$ and a line l has equation $y=4x-k$, where k is a constant.
a. Show that the x coordinate of any point of intersection between the line and the curve satisfies the equation $x^2+(2k-5)x+8+k=0$.
b. Given that the curve and the line do not intersect,
i. show that $4k^2-24k-7<0$
ii. find the set of possible values of k

14 The diagram shows a sketch of the curve with equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=(x+4)^2(x-1)$.
a. Sketch the curve C with equation $y=\mathrm{f}(x-2)$, and state the coordinates of any points where the curve C crosses or meets the x axis.
b. Write down an equation for the curve C.
c. Hence, find the coordinates of the point where the curve C crosses the y axis.



15 **a.** Sketch the curve with equation $y = x^4 - 256x^3$.
On your sketch, you should show clearly the coordinates of any points at which the curve crosses or meets the coordinate axes.
b. Hence, write down the set of values of x for which $x^4 - 256x^3 \geq 0$.

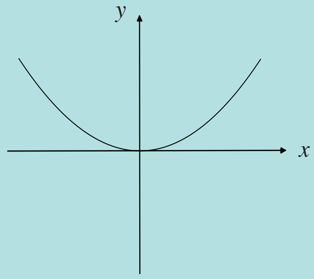
16 **a.** Given that c is a positive constant, on separate axes, sketch the graphs of
i. $y = c^2 - x^2$ **ii.** $y = x^2(x - 4c)$
On each sketch, you should show clearly the coordinates of any points at which the curve crosses or meets the coordinate axes.
b. Show that x coordinate of any point of intersection between the two curves satisfies the equation $x^3 - x^2(4c-1) - c^2 = 0$.
c. Given that the two curves meet when $x = 1$, find the exact value of c, giving your answer as simplified surd.

17 The function f is a cubic function. When $x = 1$, the function attains its maximum value of 5. When $x = 4$, the function attains its minimum value of -3.
Given that the function has only one negative root,
a. sketch the curve with equation $y = \mathrm{f}(x)$.
On your sketch, you need **not** show the coordinates of any points at which the curve crosses or meets the coordinate axes.
b. Using your graph, or otherwise, find the range of values of k for which the equation $k - \mathrm{f}(x) = 0$ has **i.** at least one solution **ii.** exactly one solution **iii.** exactly two solutions
iv. exactly three solutions **v.** no solutions
c. Find the range of values of k for which the equation $k - \mathrm{f}(x+a) = 0$ has
i. at least one solution **ii.** exactly one solution **iii.** exactly two solutions
iv. exactly three solutions **v.** no solutions
where a is a constant.
d. Find the range of values of k for which the equation $k - a\mathrm{f}(x) = 0$ has
i. at least one solution **ii.** exactly one solution **iii.** exactly two solutions
iv. exactly three solutions **v.** no solutions
where a is a constant.

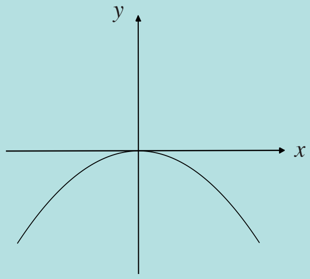
Key points from **Chapter 4**

1 A function can be thought of as a rule that is applied to an input to obtain an output.

2 Quadratics have a parabolic shape:



$y = x^2$

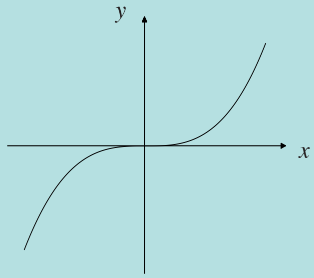


$y = -x^2$

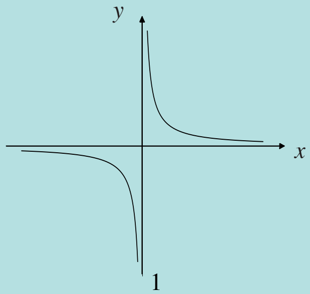
3 To find the coordinates of the turning point of a quadratic, we can use completing the square.

4 The discriminant of a quadratic tells us how many roots it has:
- if $b^2 - 4ac > 0$, then the quadratic has two distinct real roots
- if $b^2 - 4ac = 0$, then the quadratic has one repeated real root
- if $b^2 - 4ac < 0$, then the quadratic has one repeated real root

5 The graphs of the standard cubic and reciprocal function are:



$y = x^3$



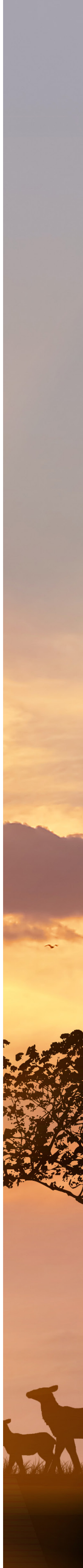
$y = \frac{1}{x}$

6 The curve with equation $y = \frac{1}{x+a} + b$ has asymptotes at $x = -a$ and $y = b$. To find the equations of the asymptotes to a curve, consider what values of x and y result in division by zero.

7 Given the graph of the function $\mathrm{f}(x)$,
- $y = \mathrm{f}(x-a)$ is a translation parallel to the x axis of a units
- $y = \mathrm{f}(x)+a$ is a translation parallel to the y axis of a units
- $y = a\mathrm{f}(x)$ is a vertical stretch by a scale factor a
- $y = \mathrm{f}(ax)$ is a horizontal stretch by a scale factor $\frac{1}{a}$

8 If a is directly proportional to b, then
1. $a \propto b$ 2. $a = kb$ for some non-zero constant k

We can find the constant of proportionality by plotting a suitable graph and finding its gradient. In this case, the graph of a against b will do.



REFRESH AND REVIEW 1

Well done! You are halfway through the course. This first refresh and review is designed to help you consolidate, unify and practise what you've learnt so far. If you are up for a challenge, try the challenge questions at the end.

1 **a.** Simplify $\sqrt{8}$.

b. Express $\dfrac{1+\sqrt{2}}{5-\sqrt{8}}$ in the form $a+b\sqrt{2}$, where a and b are integers to be found.

2 The straight line l_1 passes through the points $A(2,5)$ and the point $B(7,15)$.

a. Find the equation of the line l_1, giving your answer in the form $y=mx+c$, where m and c are constants to be found.

The line l_2 passes through the point B and is perpendicular to l_1. Given that the l_2 crosses the x axis at the point C,

b. find the coordinates of the point C.

c. Determine the equation of the circumcircle of the triangle ABC.

3 **a.** Show that the equation $x^2+kx-4=x-2k$, where k is a constant, has no real solutions if and only if $k^2-10k+17<0$.

Given that k is an integer,

b. find the set of possible values of k such that the equation $x^2+kx-4=x-2k$ has no real solutions.

4 The function f is defined such that $\mathrm{f}(x)=x^3-3x^2+4$.

a. Find the remainder obtained when f is divided by **i.** $x-1$ **ii.** $x-2$

b. Solve the equation $\mathrm{f}(x)=0$.

c. Hence, write down the solutions to the equation $\mathrm{f}(x+2)=0$.

5 On separate axes, sketch the curves

a. $y=25x^2-16x^4$

b. $y=25(x-3)^2-16(x-3)^4$

c. $y=100x^2-256x^4$

On each sketch, you should show clearly any coordinates of intersection.

6 The function f is defined such that $\mathrm{f}(x)=(2x-3)^2-6(x-2)^3+(x-3)(x-1)$.

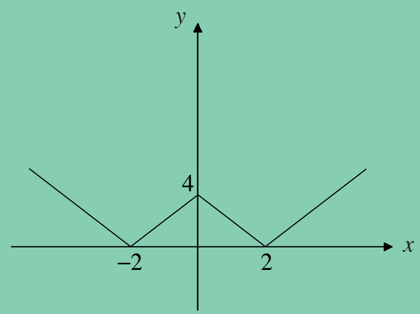
a. Express $\mathrm{f}(x)$ in the form ax^3+bx^2+cx+d, where a, b, c and d are constants to be found.

b. Show that $2x-3$ is a factor of $\mathrm{f}(x)$.

c. Express $\mathrm{f}(x)$ as a product of three linear factors.

d. Sketch the curve with equation $y=\mathrm{f}(x)$.

7 The diagram below shows a sketch of the curve with equation $y=\mathrm{f}(x)$.



The graph is linear from $(2,0)$ to $(0,4)$ and from $(0,4)$ to $(-2,0)$.

a. On separate axes, sketch the curves with equation **i.** $y=\mathrm{f}(2x)$ **ii.** $y=\mathrm{f}(x)+3$

On each sketch, you should show clearly any coordinates where the curves cross or meet the coordinate axes.

b. Find the coordinates where the curve with equation $y=\mathrm{f}(x+1)$ crosses the y axis.

c. Sketch the curve with equation $y=\mathrm{f}(x+1)$, showing clearly any coordinates of intersection.

8 **a.** Express $\dfrac{\sqrt{3^4}\times\sqrt[3]{27^2}}{\sqrt{3\sqrt{3}}}$ in the form 3^p, where p is a rational number to be found.

b. Hence, explain why

$$27<\frac{\sqrt{3^4}\times\sqrt[3]{27^2}}{\sqrt{3\sqrt{3}}}<81$$

9 **a.** Given that $3^{x-3}=9^{2x+y}$, express y in terms of x.

b. Solve the simultaneous equations

$$3^{x-3}=9^{2x+y},\ x^2+4y^2+2y=10$$

c. Interpret your solutions geometrically.

Learning goals for Chapter 10

By the end of this chapter, you should be able to:

- sketch exponential functions
- recall and apply the properties of exponential functions
- understand and use logarithmic functions
- solve equations involving logs
- apply exponentials and logs to real-life problems

Prior Knowledge

All previous chapters are useful and will be used here. In particular, you should be confident with index notation and how we can manipulate indices. Chapter 4 also develops some key skills that we will make regular use of in this chapter.

Chapter 10:
Exponentials and Logarithms

Exponential functions are a new family of functions that we will introduce you to. These are particularly useful in contexts that measure rapid growth, such as compound interest and the reproduction of bacteria. Logarithms are useful in their own way, as they are inverses to exponential functions. The picture shows an example of exponential growth, with the number of rice grains on each tile quickly increasing with the number of squares.

10.1 Exponential functions and their graphs

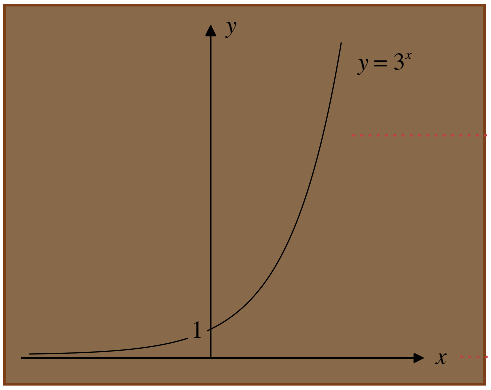
Exponential functions are of the form $y = a^x$, where a is a positive constant.

Worked Example 1

Sketch the curve with equation $y = 3^x$. On your sketch, you should show clearly the coordinates of any points where the curve crosses or meets the coordinate axes.

!BREAK IT DOWN!

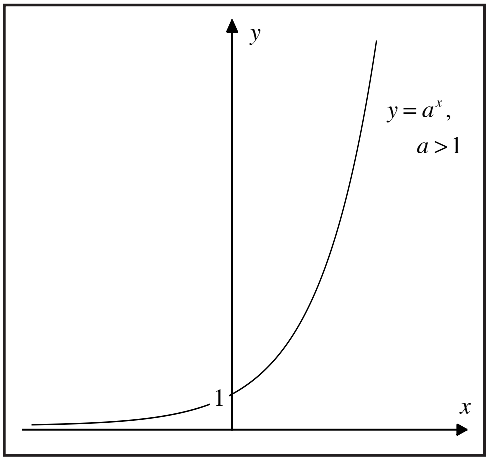
- We don't know what this function looks like just yet, but we can reason logically:
  - as x gets large and positive, the values of the function get large very quickly. *The function tends to positive infinity as x tends to positive infinity.*
  - when x is negative, the values of the function are less than 1. As x tends to negative infinity, the values continue to get even smaller. *So the function tends to zero as x tends to negative infinity.*
  - the function is never negative, so is always above the x axis. When $x = 0$, the value of the function is 1.



As x gets large and positive, the values of the function get arbitrarily large.

As x gets small and negative, the function tends to 0. The x axis is an **asymptote** to the curve.

In fact, our logic from Worked Example 1 is correct for all values of a greater than 1. In other words:



2 **a.** Sketch the curve with equation $y = x(3-x)^2$.
On your sketch, you should show clearly the coordinates of any points where the curve crosses or meets the coordinate axes.
b. Use your answer to part **a** to sketch, on separate axes, the gradient function of the curve in **a**.
c. Find the gradient function of the curve in **a** by differentiation.
d. Hence, find the coordinates of any stationary points on the curve in **a**. Explain what this information tells you about the curve in **b**.
e. By plotting the gradient function you found in **c**, verify your answer to part **b**.

11.10 Applications

Since differentiation tells us information about the rate of change of a variables with respect to another, it has many applications. We already saw it being used in problems about exponential growth and decay in Chapter 10, now we will look at some more.

Worked Example 18

The area, A cm^2, of an expanding sphere of radius r cm is given by $A = 4\pi r^2$. Find the rate of change of the area of the sphere when the radius is 4 cm.

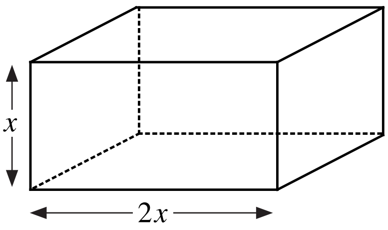
$$\frac{dA}{dr} = 8\pi r \Rightarrow \left.\frac{dA}{dr}\right|_{r=4} = 8\pi(4) = 32\pi$$

Differentiate with respect to r and then sub in $r = 4$.

Worked Example 19

An open rectangular box is in the shape of a cuboid and has total surface area 48 m^2. The length of the cross-section of the box is $2x$ m and the width of the cross-section of the box is x m.

a. Show that the volume V m^3 of the box is $V = 24x - 2x^3$.
b. Given that x varies, use calculus to find the maximum or minimum of V.
c. Using further calculus, determine whether the value of V you found is a maximum or minimum. Justify your answer.



!BREAK IT DOWN!

- We don't know the length of the box, so we can't find the volume by just using the formula volume = area of cross section × length.
- We are given the area of the box, so we can use that to form another equation and hence eliminate the length of the box from the volume formula.

$$\begin{aligned} \textbf{a.}\ \text{Area} &= 2(2x)(x) + 2(x)(y) + 1(2x)(y) \\ &= 4x^2 + 4xy \end{aligned}$$

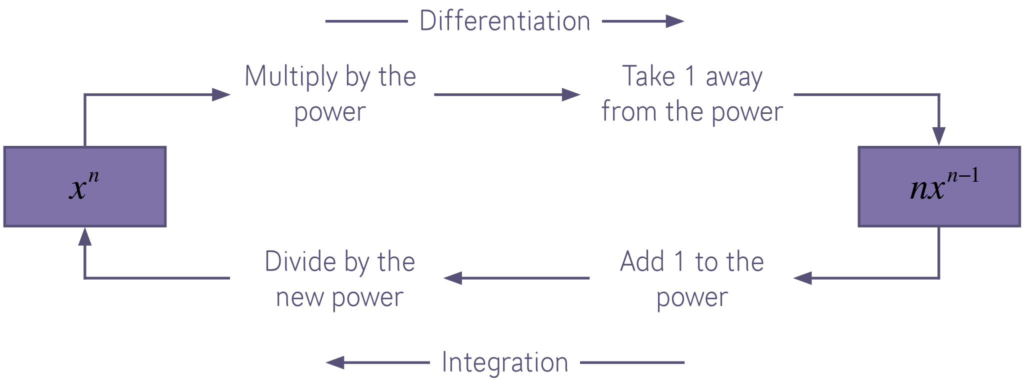
$$A = 48 \Rightarrow 48 = 4x^2 + 4xy$$

$$\Rightarrow y = \frac{12 - x^2}{x}$$

Let y be the length of the box. Find out the total area of the box and set it equal to 48. Then re-arrange for y. When working out the area of the box, remember that the box is open, so its upper surface should not feature in the calculation.

12.2 Integrating x^n

Integration is the reverse process of differentiation. Since we already know how to differentiate x^n, this fact means that we can figure out what its integral is with relative ease.



So, in general, $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c$.

Worked Example 1

Find $\int x^3 \, dx$.

$$\int x^3 \, dx = \frac{x^4}{4} + c$$

Add 1 to the power (3) and divide by the new power (4). Remember to include a constant of integration.

Worked Example 2

Find $\int \sqrt{x} \, dx$.

$$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx$$

Re-write the integrand in index form.

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

Add 1 to the power (1/2) and divide by the new power (3/2). Remember to include a constant of integration.

$$= \frac{2}{3} x^{\frac{3}{2}} + c$$

Simplify. Remember:

$$\frac{1}{\frac{3}{2}} = \frac{1}{1} \div \frac{3}{2} = \frac{1}{1} \times \frac{2}{3} = \frac{2}{3}$$

Exercise 12A

1 Integrate the following expressions with respect to x.

a. x^2

b. $-x^{-3}$

c. $2x^{\frac{3}{2}}$

d. $x^{-\frac{1}{3}}$

e. $7x^{-\frac{4}{3}}$

f. $4\sqrt{x^5}$

g. $x\sqrt{x}$

h. $x\sqrt{x^3}$

i. $\sqrt{x}\sqrt[3]{x^2}\sqrt[4]{x^3}$

j. $\dfrac{5x^3}{\sqrt{x}}$

k. 10

l. $\sqrt{x^{-4}}$

2 **Discussion question:** try integrating x^{-1} with respect to x. Why doesn't the formula we have been using so far not work for this case?
(This is a more difficult integral. You will learn about it at A2. When we integrate it, we get the natural logarithm.)

12.3 Integrating expressions with more than one term

Just as we can differentiate an expression term-by-term, we can also integrate an expression term-by-term. Exercise 11**A** asked you to integrate 10 with respect to x. We can think about this in two ways. If we use the formula, note that $10 = 10x^0$. To integrate this, we add one to the power and divide by the new power to get $10x + c$. Alternatively, you could think it about it by recalling that when we differentiate $10x$ with respect to x, we get 10; integration just reverses this.

Worked Example 3

Find $\int \left(3x^2 - \sqrt{x} + 5\right) dx$.

Integrand - the expression to be integrated.

$$\int \left(3x^2 - \sqrt{x} + 5\right) dx = \int \left(3x^2 - x^{\frac{1}{2}} + 5\right) dx$$

Re-write the integrand in index form.

$$= \frac{3x^3}{3} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5x + c$$

Integrate the expression term-by-term.

$$= x^3 - \frac{2}{3}x^{\frac{3}{2}} + 5x + c$$

Simplify.

CM

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